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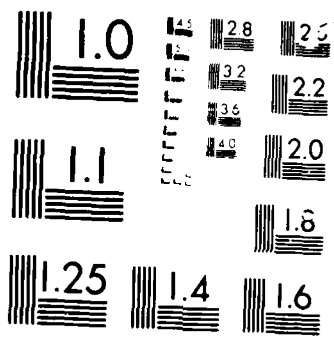
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FINAL REPORT ON NONLINEAR WAVES

OFFICE OF NAVAL RESEARCH GRANT #N00014-76-C-0867

BY

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and

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The central theme involved in this work is the continuing study of certain fundamental features associated with the nonlinear wave propagation arising in and motivated by physical problems. The usefulness of the work is attested to by the varied applications, and wide areas of interest in physics, engineering and mathematics. The work accomplished involves wave propagation in a number of areas including fluid mechanics, plasma physics, theoretical physics, statistical mechanics, nonlinear optics, multidimensional solitons, multidimensional inverse problems, Painleve equations, direct linearizations of certain nonlinear wave equations, DBAR problems, Riemann-Hilbert boundary value problems, algebraic methods and symmetry analysis of multidimensional systems, differential geometry, etc. Of particular interest to the Navy is the recent discovery that many of the equations describing ship hydrodynamics in channels of finite depth obey nonlinear equations which have been studied extensively by our group.

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The central theme involved in this work is the continuing study of certain fundamental features associated with the nonlinear wave propagation arising in and motivated by physical problems. The usefulness of the work is attested to by the varied applications, and wide areas of interest in physics, engineering and mathematics. The work accomplished involves wave propagation in a number of areas including fluid mechanics, plasma physics, theoretical physics, statistical mechanics, nonlinear optics, multidimensional solitons, multidimensional inverse problems, Painleve equations, direct linearizations of certain nonlinear wave equations, DBAR problems, Riemann-Hilbert boundary value problems, algebraic methods and symmetry analysis of multidimensional systems, differential geometry, etc. Of particular interest to the Navy is the recent discovery that many of the equations describing ship hydrodynamics in channels of finite depth obey nonlinear equations which have been studied extensively by our group.

(1) Research Objectives

The continuing aspects of the work performed under this grant has been the study of the nonlinear wave phenomena associated with physically significant systems. As mentioned above, this work has important applications in fluid dynamics (e.g. long waves in stratified fluids, solitons generated by ships), nonlinear optics (e.g. self-induced transparency, and self-focussing of light), and mathematical physics as well as important consequences in mathematics. Individuals working with us and hence partially associated with this grant include: Dr. Peter Clarkson, Postdoctoral Research Associate in Mathematics and Computer Science,

Dr. Daniel Bar Yaacov, Postdoctoral Research Associate in Mathematics and Computer Science, Mr. Ugurhan Mugan, a graduate student in Mathematics and Computer Science, Mr. Vassilis Papageorgiou, a graduate student in Mathematics and Computer Science and Mr. Rogelio Balart, a graduate student in Mathematics and Computer Science. Recent publications supported by this research grant are enclosed.

Areas of Study Include:

Solutions of nonlinear multidimensional systems arising in Physics

Inverse problems, especially in multidimensions and DBAR methodology

Riemann-Hilbert boundary value problems and inverse problems

Solitons in multidimensional systems, solitons generated by ships in narrow channels

IST for nonlinear singular integro-differential equations; the Benjamin-Ono equation, the intermediate Long Wave Equation, the Sine-Hilbert equation, multidimensional generalizations.

Discrete IST and numerical simulations

Painleve equations

Focussing singularities in nonlinear wave propagation

Applications to surface waves, internal waves, shear flows, nonlinear optics, S.I.T., relativity etc.

Direct linearizing methods for nonlinear evolutions equations

Multidimensional generalizations of the Sine-Gordon and wave equations arising in differential geometry

Algebraic methods and symmetries of multidimensional nonlinear evolution equations

Solutions to semiperiodic multidimensional equations.

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MULTIDIMENSIONAL NONLINEAR EVOLUTION EQUATIONS AND INVERSE SCATTERING

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In this paper we will review some recent work done in the field of integrable nonlinear evolution equations and inverse scattering. We will concentrate on the basic underlying areas and refer interested readers to suitable references for complete details; specifically background material can be found in various texts on this subject (e.g. [1] by Ablowitz and Segur). More recent references will be given as necessary. The outline of the paper is as follows.

1) Introductory remarks.

2) A discussion of two separate but related issues. Namely, (a) solving certain nonlinear evolution equations in infinite space; and (b) inverse scattering. These are important problems having many physical applications. Moreover, they are related to each other by what we refer to as the Inverse Scattering Transform (IST).

3) At the end of the paper we will make some remarks on the possibility of solving nonlinear evolution equations in high dimensions (i.e. equations with more than two spatial and one time variable) by using the IST method as we now understand it.

1. Introduction

The prototype nonlinear evolution equations for our purposes will be the Korteweg-deVries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

in one spatial dimension, and the Kadomtsev-Petviashvili (KP) equation

$$(u_t - 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{2}$$

in two spatial dimensions. (It turns out that the sign of σ^2 is critical: there being two cases labeled by $\mathrm{KP_I}$: $\sigma^2 = -1$; $\mathrm{KP_{II}} = \sigma^2 = 1$.)

Historically speaking, the KdV equation was the first equation solved (on the infinite line) by use of inverse scattering. Subsequently numerous other equations of physical interest in one spatial dimension were solved e.g. *nonlinear Schrödinger, sine-Gordon, three-wave interaction*, modified KdV, Boussinesq, ... These equations are all partial differential equations. In fact, there are other equations which are discrete in space and continuous in time (differential-difference) and equations discrete in both space and time which also may be solved by IST. One other class of equations in one spatial and one time dimension fit into this scheme, namely nonlinear singular integro-differential equations; with the prototype being the so-called

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Intermediate Long Wave equation [2a],

$$u_t - \frac{1}{\delta}u_x + 2uu_x + (Tu)_{xx} = 0, \quad Tu = \frac{1}{2\delta}\int_{-\infty}^{\infty} \coth\frac{\pi}{2\delta}(\xi - x)u(\xi)\,d\xi. \tag{3}$$

As $\delta \to 0$, (3) tends to the KdV equation (with appropriate coefficients) and as $\delta \to \infty$ it tends to the so-called Benjamin–Ono equation

$$u_t + 2uu_x + (Hu)_{xx} = 0, \quad H_u = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x}\,d\xi. \tag{4}$$

The method to solve (4) was recently found and it has certain features in common with some two-dimensional problems – specifically KP_I (see [2b]).

It should also be remarked that some ode's can also be solved by similar methods; specifically the classical equations of Painlevé (see for example [3]). We will not dwell on this aspect any further in this lecture.

In two spatial one time dimension the KP equation is only one of the equations that can be solved in infinite space. However, an effective method was not realized until a short time ago. The important new idea of treating inverse scattering as a "$\bar\partial$ problem" (see [9a]) was used in [4] to solve KP_{II} and paved the way for the development of the IST for a wide class of equations in 2 + 1 dimensions (a review of this and related work can be found in [5a, b]). It should be mentioned that earlier work on KP_I had been done by Manakov [6a] and more recently by Fokas and Ablowitz [6b] and on the multidimensional three-wave equation by Cornille [7a] and Kaup [7b]. KP_{II} and others like it depart significantly from previous work and its study has led us to develop a general method to do inverse scattering in n spatial dimensions as we will indicate in this review (see [8a, b, c]).

The concept of treating inverse scattering as a "$\bar\partial$ problem" was originally discussed by Beals and Coifman in their study of first order systems of differential equations [9a]. Beals and Coifman have also recently considered multidimensional inverse scattering via $\bar\partial$ methods [9b].

It should be noted that important contributions in the study of multidimensional inverse scattering associated with the time-independent Schrödinger problem have been made by Faddeev [10] and Newton [11]. In one dimension we also note the important contributions of Shabat [12a], Mikhailov [12b] and Caudrey [12c]. Some of the work in this review is related to these studies although the methodology is different.

2. Inverse scattering and ther inverse scattering transform

The method of solution by IST begins with the study of two compatible linear operators (Lax pairs) (L depends on one or more "potentials" or functions which we call u)

$$Lv = \lambda v, \tag{5}$$

$$v_t = Mv, \tag{6}$$

connected by the compatibility condition

$$L_t - [L, M] = 0. \tag{7}$$

when the flow is isospectral, $\lambda_t = 0$. (7) is the nonlinear evolution equation to be solved. λ is a spectral parameter, which as it turns out loses significance in spatial dimensions greater than one. L is a spatial operator only; with time acting as a parameter. The parametric dependence in time is what allows us to study the question of inverse scattering separately and then after this task is completed allows us to solve the relevant nonlinear equation (7). For KdV the operators are

$$L = \frac{\partial^2}{\partial x^2} - u, \quad M = (4\lambda + 2u)\frac{\partial}{\partial x} - u_x. \tag{8}$$

The reader can now verify that (7) yields (1). It should be noted that there are generalizations of (5)–(7), but we shall not be concerned with that here.

The direct (or forward) scattering problem associated with L means given a potential, in a desired function class, and solve for eigenfunctions corresponding to suitable initial or boundary conditions. Usually, appropriate eigenfunctions are defined in terms of an integral equation (e.g. via Green's functions). From the eigenfunctions scattering coefficients, eigenvalues, etc. can be calculated. Call the set of all such data obtainable from the solution of (5) $\hat{S}$.

The inverse problem is as follows. Given some subset S of $\hat{S}$ (i) reconstruct the eigenfunctions and the potential; (ii) characterize the analytical, algebraic, and/or topological constraints on the data in order to find a potential in the desired function class.

In recent years significant strides forward have been made in regard to the solution of those inverse problems motivated by the study of nonlinear evolution equations. Examples in one dimension are

(i) $\dfrac{\mathrm{d}^n v}{\mathrm{d}x^n} + \sum_{j=2}^{n} u^{(j)}(x)\dfrac{\mathrm{d}^{n-j}v}{\mathrm{d}x^{n-j}} = \lambda v$, $\quad u^{(j)}(x), v(x, \lambda)$ scalar [see 9c];

(ii) $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{i}\lambda Jv + qv$, $\quad v(x, \lambda), q(x) \in \mathbf{C}^{N\times N}$, $\quad J = \mathrm{diag}(J^1, \dots, J^N), (J^i \neq J^j, i \neq j)$ [see 9d].

In multidimensions examples are

(iii) $\sigma\dfrac{\partial v}{\partial y} + \Delta v - u(x, y)v = 0$, $\quad \sigma = \sigma_R + \mathrm{i}\sigma_I$, $\quad x \in \mathbb{R}^n$, $\quad y \in \mathbb{R}$, $\quad \Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ [see 8a, 8c, 9b];

(iv) $-\Delta v + u(x)v = \lambda v$ [see 10, 11, 8a, 8c, 9c];

(v) $\dfrac{\partial v}{\partial y} + \sigma \sum_{i=1}^{n} J_i \dfrac{\partial v}{\partial x_i} = qv$, $\quad \sigma = \sigma_R + \mathrm{i}\sigma_I$, $\quad x \in \mathbb{R}^n$, $\quad y \in \mathbb{R}$; $\quad v, q \in \mathbf{C}^{N\times N}$, $\quad J_i = \mathrm{diag}(J_i^1, \dots, J_i^N)$,

$(J_i^i \neq J_i^j, i \neq j)$ [see 8b].

The inverse problem for (i) and (ii) may be written in a compact form. Namely solve

$$(\mu_+ - \mu_-)(x, k) = \mu_-(x, \alpha(k))V(x, k)$$

on Σ (Σ is an appropriate contour in the complex k-plane and V is a function depending explicitly on the scattering data and $\alpha(k)$ is problem dependent) with

$$\mu_{\pm} \xrightarrow[|k| \to \infty]{} I, \quad \alpha(k), V(x, k) \text{ given on } \Sigma.$$

and

$$\mu_{\pm}(x,k) \text{ meromorphic in } k \in \mathbf{C}/\Sigma. \tag{9}$$

$\mu_{\pm}(x,k)$ has a finite number of poles with locations specified: $k_1,\ldots,k_n$; and $\operatorname{Res}_{k=k_j}\mu_{\pm}(x,k)$ specified appropriately.

In (9), $\mu(x,k)$ is associated with an eigenfunction of the given operator. It is related to $v(x,k)$ by

$$v(x,k) = \mu(x,k)\,e^{\theta_{L_0}(x,k)},$$

where $\theta_{L_0}(x,k)$ is a concrete phase factor which depends on the unperturbed (potential zero) operator. The parametric dependence $\lambda = \lambda(k)$ is explicitly given (chosen for convenience).

(9) is a variant of the usual Riemann–Hilbert factorization problem. The standard situation involves finding $\mu_{\pm}$ analytic off Σ without any extra parameter such as x.

Corresponding to (i) and (ii) above, the second order case is classical and has been studied by numerous authors (a review of this appears in [1]). Although some work had been done for third order scalar operators nevertheless it has only been within the past few years that the solution to the general nth order case has been found. It should be noted that the matrix system (ii) above has also been studied in [12a–c]. A thorough analysis of the problems, including the case of complex diagonal elements of J appears in [9d].

To be concrete we shall given the results for the inverse problem associated with the one-dimensional time-independent Schrödinger equation: i.e. (i) above with $n=2$, $u(x) = -u^{(2)}(x)$. Let $\lambda(k) = -k^2$, then the scattering equation is

$$v_{xx} + (k^2 - u)v = 0, \quad -\infty < x < \infty, \quad v = \mu\,e^{-ikx}, \tag{10}$$

$$\mu_{xx} - 2ik\mu_x - u\mu = 0. \tag{11}$$

The relevant function class for $u(x)$ is $\int_{-\infty}^{\infty}(1+|x|)|u|\,dx < \infty$. $v(x,k)$ has solutions (Jost functions) which we denote by

$$\phi(x,k) \underset{x\to-\infty}{\simeq} e^{-ikx}, \qquad \left.\begin{array}{l} \bar{\psi}(x,k) \underset{x\to+\infty}{\simeq} e^{-ikx}, \\ \psi(x,k) \underset{x\to+\infty}{\simeq} e^{ikx}. \end{array}\right\} \tag{12a}$$

Functions with "nice" analytical properties are obtained by multiplying by a suitable exponential factor:

$$M(x,k) = \phi\,e^{ikx} \underset{x\to-\infty}{\simeq} 1, \qquad \left.\begin{array}{l} \bar{N}(x,k) = \bar{\psi}\,e^{ikx} \underset{x\to+\infty}{\simeq} 1, \\ N(x,k) = \psi\,e^{ikx} \underset{x\to+\infty}{\simeq} e^{2ikx}. \end{array}\right\} \tag{12b}$$

The relationship

$$\psi(x,k) = \bar{\psi}(x,-k) \tag{12c}$$

implies

$$N(x,k) = \bar{N}(x,-k)\,e^{2ikx}. \tag{12d}$$

Completeness of these eigenfunctions requires

$$M(x,k) = a(k)\bar{N}(x,k) + b(k)N(x,k),$$

or, using (12d),

$$\frac{M(x,k)}{a(k)} = \bar{N}(x,k) + r(k)\,e^{2ikx}\bar{N}(x,-k), \tag{12e}$$

where $r(k) = b(k)/a(k)$. The analyticity of $M(x,k)$, $\bar{N}(x,k)$ is deduced by studying the following integral equations:

$$M(x,k) = 1 + \int_{-\infty}^{\infty} G_+(x-x',k)u(x')M(x',k)\,\mathrm{d}x', \tag{12f}$$

$$\bar{N}(x,k) = 1 + \int_{-\infty}^{\infty} G_-(x-x',k)u(x')\bar{N}(x',k)\,\mathrm{d}x', \tag{12g}$$

where

$$G_{\pm}(x,k) = \frac{1}{2\pi}\int_{C_{\pm}} \frac{e^{i\xi x}}{\xi(\xi-2k)}\,\mathrm{d}\xi, \tag{12h}$$

$C_{\pm}$ being the contour below $(+)$/above $(-)$ the singularities $\xi = 0$, $\xi = 2k$ inside the integral (12h). $G_{\pm}(x,k)$ is analytic for $\operatorname{Im} k \gtrless 0$ and vanishes as $|k| \to \infty$. $M(x,k)$, $\bar{N}(x,k)$ are therefore analytic for $\operatorname{Im} k > 0$, $\operatorname{Im} k < 0$ respectively and tend to unity as $|k| \to \infty$.

The scattering coefficient $a(k)$ is also analytic for $\operatorname{Im} k > 0$ and tends to unity as $|k| \to \infty$ (this can be deduced from the fact that $a(k)$ is a Wronskian of M, N). $a(k)$ can vanish at a finite number of locations in the upper half plane: $k = k_1, \ldots, k_n$, $\operatorname{Im} k > 0$. Calling

$$\mu_+(x,k) = \frac{M(x,k)}{a(k)}, \quad \mu_-(x,k) = \bar{N}(x,k), \tag{12i}$$

we see that (9e) is a special case of (9) where $\alpha(k) = -k$, $V(x,k) = r(k)e^{2ikx}$. The appropriate residue statement is

$$\operatorname*{Res}_{k=k_j}(\mu_+(x,k)) = c_j\,e^{2ik_jx}\mu_-(x,k_j), \tag{12j}$$

C_j being called the normalization constants.

It is worthwhile noting that when no poles (i.e. no eigenvalues or boundstates) appear, then the solvability of (12e) follows from the work of Gohberg and Krein [13] in which they prove the existence of uniqueness of the solution of the corresponding Riemann–Hilbert factorization problem (in a generic sense).

For completeness we list the integral equations for the eigenfunction and potential reconstruction:

$$N(x,k) = e^{2ikx}\left(1 - \sum_{j=1}^{n} \frac{c_j N_j(x)}{k - k_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\xi) N(x,\xi)}{\xi + k + i0}\,d\xi\right), \tag{12k}$$

$$N_l(x) = e^{2ik_l x}\left(1 - \sum_{j=1}^{n} \frac{c_j N_j(x)}{k_l + k_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\xi) N(x,\xi)}{\xi + k_l}\,d\xi\right), \tag{12l}$$

$$u(x) = \frac{\partial}{\partial x}\left(\sum_{j=1}^{n} 2ic_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} r(k) N(x,k)\,dk\right). \tag{12m}$$

The solution of the initial value problem for suitably decaying functions $u(x,k)$ of KdV is obtained by noting that $r(k,t) = r(k,0)e^{8ik^3t}$. This follows from the second linear operator M: see (6), (8). The reconstruction of $u(x,t)$ then follows from the inverse problem. In the general case, the data $V(x,k,t)$ in (9) also evolves simply in time (e.g. $V(x,k,t) = V(x,k,0)e^{\omega(k)t}$ when V, ω are scalars). Schematically, we have:

$$\begin{array}{c}
\text{(Direct problem)} \qquad\qquad\qquad \text{(From } M \text{ operator)} \quad \text{(From inverse problem)} \\
\searrow \qquad\qquad\qquad\qquad\qquad\qquad \searrow \qquad \swarrow \\
u(x,0) \to \mu_{\pm}(x,k,t=0) \to V(x,k,0) \to V(x,k,t) \to \mu_{\pm}(x,k,t) \to u(x,t)
\end{array}$$

The method of solution is what is usually referred to as the Inverse Scattering Transform: IST. This program has been carried out for a surprisingly large number of physically interesting equations in one spatial dimension. In fact, the only equation in one spatial dimension mentioned above that does not have an associated inverse problem of the form (9) is the Benjamin–Ono equation (4). It shares with the KP$_1$ equation an inverse problem of the nonlocal R–H form:

$$(\mu_+ - \mu_-)(x,k) = \int \mu_-(x,k')V(x,k',k)\,dk'. \tag{13}$$

Next, we shall discuss the KP equation and its associated scattering operator L,

$$\sigma v_y + v_{xx} + u(x,y)v = 0. \tag{14}$$

Note in (14) we have taken the eigenvalue $\lambda = 0$ without loss of generality (by the scaling property of v). Since the analysis for the generalization

$$\sigma v_y + \Delta v + u(x,y)v = 0, \tag{15}$$

where $\sigma = \sigma_R + i\sigma_I$, $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, is a natural extension of that in two dimensions, we shall discuss this case. Scattering parameters arise in (15) by looking for a function $\mu = \mu(x,y,k)$ where

$$v = \mu\, e^{ik\cdot x - k^2 y/\sigma}, \tag{16}$$

$$\sigma\mu_y + \Delta\mu + 2ik\cdot\nabla\mu + u\mu = 0, \tag{17}$$

and $k = k_R + ik_I \in \mathbb{C}^n$. We shall consider $\sigma_R = 0$, $\sigma_R < 0$.

We look for a solution $\mu(x, y, k)$ bounded for all x, y and $\mu \to 1$ as $|k| \to \infty$. The latter condition is a convenient normalization. If we should consider (17) for $\sigma = \pm 1$ in analogy to the KP_{II} scattering problem, we immediately notice that the dominant operator is the heat operator which is illposed as an initial value problem. Even though we pose a boundary problem, immediately we are led to believe that in this case there will be some type of unusual behavior. In fact in refs. [4, 8a] it is shown that the bounded function μ for $\sigma_R \neq 0$ may be analytic nowhere as a function of k. Specifically $\mu = \mu(x, y, k_R, k_I)$. In particular μ is constructed from the following equation. Given $u(x, y) \to 0$ sufficiently rapidly at ∞, the direct problem is

$$\mu = 1 + \tilde{G}(u\mu), \tag{18}$$

where

$$\tilde{G}f \equiv G * f \equiv \iint G(x - x', y - y', k_R, k_I) f(x', y')\, dx'\, dy'. \tag{19}$$

The Green's function G is obtained from

$$G(x, y, k_R, k_I) = C_{n+1} \iint \frac{e^{i(x\cdot\xi - y\eta)}}{i\sigma\eta - \xi^2 - 2k\cdot\xi}\, d\xi\, d\eta, \quad C_n \equiv \frac{1}{(2\pi)^n}, \tag{20a}$$

$$= \frac{\operatorname{sign}(y)}{\sigma} C_n \int e^{(i/\sigma)y(\xi^2 - 2k\cdot\xi) - ix\cdot\xi}\, \Theta\left[-y\sigma_R\left(\xi^2 - 2\left(k_R + \frac{k_I\sigma_I}{\sigma_R}\right)\cdot\xi\right)\right] d\xi, \tag{20b}$$

where $\Theta(x) = \{1 \text{ for } x > 0,\ 0 \text{ for } x < 0\}$. In constructing (20) we have looked for a bounded Green's function, and have taken the Fourier transform in both x and y.

Taking the $\bar{\partial}$ derivative of (18) with respect to $\bar{k}$ we find ($\partial/\partial\bar{k} = \frac{1}{2}(\partial/\partial k_R + i\partial/\partial k_I)$):

$$\frac{\partial\mu}{\partial\bar{k}} = \frac{\partial\tilde{G}(u\mu)}{\partial\bar{k}} + \tilde{G}\left(u\frac{\partial\mu}{\partial\bar{k}}\right). \tag{21}$$

The first term in (21) is calculated directly using the definition of the Green's function (20).

$$\frac{\partial\tilde{G}(u\mu)}{\partial\bar{k}} = -\frac{C_n}{i\sigma_R} \int e^{i\beta(x, y, k_R, k_I, \xi)} T(k_R, k_I, \xi)(\xi - k_R)\delta(s(\xi))\, d\xi, \tag{22a}$$

where

$$T(k_R, k_I, \xi) \equiv \iint e^{-i\beta(x, y, k_R, k_I, \xi)} u(x, y)\mu(x, y, k_R, k_I)\, dx\, dy, \tag{22b}$$

$$\beta(x, y, k_R, k_I, \xi) = \left(x - 2y\frac{k_I}{\sigma_R}\right)\cdot(\xi - k_R), \tag{22c}$$

$$s(\xi) = s(\xi, k_R, k_I) \equiv \left(\xi - \frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R + \frac{\sigma_I}{\sigma_R}k_I\right)^2 \tag{22d}$$

and $\delta(x)$ is the usual Dirac delta function. One can derive (22) either by taking the $\bar{\partial}$ derivative directly on

(20b) or on (20a) using the well-known fact

$$\frac{\partial}{\partial \bar{k}}\left(\frac{1}{\bar{k}-k_0}\right)=\pi\delta(k-k_0). \tag{22e}$$

From (22) one can readily calculate $\partial\mu/\partial\bar{k}_j$ (assuming (18) has no homogeneous solutions).

$$\frac{\partial\mu}{\partial\bar{k}_j}=\tilde{T}_j\mu=-\frac{C_n}{|\sigma_R|}\int e^{i\beta(x,y,k_R,k_I,\xi)}T(k_R,k_I,\xi)(\xi_j-k_R)\delta(s(\xi))\mu(x,y,\xi,k_I)\,d\xi. \tag{23}$$

(23) is found by noting that $\partial\mu/\partial\bar{k}_j$ is a suitable superposition over a fundamental solution $W(x,y,k_R,k_I,\xi)$ satisfying

$$W(x,y,k_R,k_I,\xi)=e^{i\beta(x,y,k_R,k_I,\xi)}+\tilde{G}(uW). \tag{24}$$

Using the symmetry condition on the Green's function.

$$e^{-i\beta(x,y,k_R,k_I,\xi)}G(x,y,k_R,k_I)=G(x,y,\xi,k_I), \quad \text{on} \quad s(\xi)=0, \tag{25}$$

allows us to find

$$W(x,y,k_R,k_I,\xi)=e^{i\beta(x,y,k_R,k_I,\xi)}\mu(x,y,\xi,k_I), \quad \text{on } s(\xi)=0, \tag{26}$$

and then (23) follows.

A special case of (23) is $n=1$ whereupon $\partial\mu/\partial\bar{k}_j$ depends locally on μ. For $n=1$, let $k_1\equiv k$; then (23) reduces to

$$\frac{\partial\mu}{\partial\bar{k}}=\frac{C_1}{|\sigma_R|}\,\mathrm{sgn}\left(k_R+\frac{\sigma_I}{\sigma_R}k_I\right)e^{i\beta(x,y,k_R,k_I,\xi_0)}T(k_R,k_I,\xi_0)\mu(x,y,\xi_0,k_I). \tag{27}$$

where $\xi_0=-k_R-(2\sigma_I/\sigma_R)k_I$. (27) is relevant to the solution of KP: KP_{II}: $\sigma_I=0$, $\sigma_R=-1$ (see [4]) and KP_I $\sigma_I=1$, $\sigma_R\to 0$ ($\sigma_R<0$) with the scaling $\hat{k}_I=k_I/\sigma_R$ (also see the discussion of the limit to the time-dependent Schrödinger equation later in this paper).

The above discussion is entirely within the context of the direct scattering problem. However, it suggests what the natural data might be for this problem. We shall call $T(k_R,k_I,\xi)$ the inverse data.

The inverse problem is: given $T(k_R,k_I,\xi)$ construct $u(x,y)$. However, it is immediately transparent that there is a serious redundancy question. Namely $T(k_R,k_I,\xi)$ is a function of $3n$ parameters with one restriction (the restriction is due to $\delta(s(\xi))$ in (23); i.e. T will be given as a function of $3n-1$ variables and we wish to construct a function $u(x,y)$ depending on $n+1$ variables. But for $n=1$, namely for the problem in two spatial dimensions the difficulty disappears. As (27) shows $T=T(k_R,k_I,\xi_0(k_R,k_I))$, hence T is a function of two parameters as is u.

Using (23) there are numerous reconstruction formulae for u available. However, serious restrictions on T must be imposed in order to obtain a function u depending only on x, y and vanishing at ∞. This is part of the characterization question, i.e. which inverse data $T(k_R,k_I,\xi)$ are "admissible".

One set of inversion formulae for μ is obtained from the generalized Cauchy formula

$$\mu(k)=\frac{1}{2\pi i}\oint_C\frac{\mu(l)}{l-k}\,dl+\frac{1}{\pi}\iint_R\frac{\partial\mu/\partial\bar{l}}{k-l}\,dl_R\,dl_I. \tag{28}$$

(Another, more symmetric inversion uses the Bochner–Martinelli formula but this is outside the scope of the present review.) Applying this to our problem where $\mu \to 1$, $|k| \to \infty$ (the first term is unity) we have

$$\mu(x, y, k_{\mathrm{R}}, k_{\mathrm{I}}) = 1 + \frac{1}{\pi} \iint \frac{\dfrac{\partial \mu}{\partial \bar{k}_j}(x, y, k'_{\mathrm{R}}, k'_{\mathrm{I}})}{k_j - k'_j} \, \mathrm{d}k'_{\mathrm{R}_j} \, \mathrm{d}k'_{\mathrm{I}_j}. \tag{29}$$

where we use the simplified notation $k'_{\mathrm{R}} \equiv (k_{\mathrm{R}_1}, \ldots, k'_{\mathrm{R}_j}, \ldots, k_{\mathrm{R}_n})$ and similarly for k'_{I}. (29) is a linear integral equation for (using 23)) the potential is constructed from

$$u(x, y) = \frac{2\mathrm{i}}{\pi} \frac{\partial}{\partial x_j} \iint \frac{\partial \mu}{\partial \bar{k}_j}(x, y, k'_{\mathrm{R}}, k'_{\mathrm{I}}) \, \mathrm{d}k'_{\mathrm{R}_j} \, \mathrm{d}k'_{\mathrm{I}_j}. \tag{30}$$

(30) is obtained by taking $k_j \to \infty$ in (18) and (29) and comparing the results.

It is clear that in general the right-hand side of (30) will be a function of $k_{\mathrm{R}_i}, k_{\mathrm{I}_i}$; $i = 1, 2, \ldots, j-1, j+1, \ldots, n$. One possible way of characterizing admissible data would be to require $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ to be such that the RHS of (30) be independent of these parameters, for all j. Such a requirement is analogous to what Newton refers to as the "miracle" in the time-independent problem (see [11]). However, in this formulation we can go further and give conditions directly on $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$. The importance of characterizing $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ directly not only has to do with understanding on which manifolds of $k_{\mathrm{R}}, k_{\mathrm{I}}, \xi$ can one hope to reconstruct the potential, but also may indicate how one could in principle measure data so as to produce local potentials in a stable manner.

For $n > 1$ the compatibility condition $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$ $(i \neq j)$ leads to a nontrivial restriction on T; one which is nonlinear:

$$\mathscr{L}_{ij}(T) = N_{ij}(T), \tag{31a}$$

where

$$\mathscr{L}_{ij} = (\xi_j - k_{j_{\mathrm{R}}})\left(\frac{\partial}{\partial \bar{k}_i} + \frac{1}{2}\frac{\partial}{\partial \xi_i}\right) - (\xi_i - k_{i_{\mathrm{R}}})\left(\frac{\partial}{\partial \bar{k}_j} + \frac{1}{2}\frac{\partial}{\partial \xi_j}\right), \tag{31b}$$

$$N_{ij}[T](k, \xi) = \int \left[(\xi'_j - k_{j_{\mathrm{R}}})(\xi_i - \xi'_i) - (\xi'_i - k_{i_{\mathrm{R}}})(\xi_j - \xi'_j)\right] \delta(s(\xi')) T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi') T(\xi', k_{\mathrm{I}}, \xi) \, \mathrm{d}\xi'. \tag{31c}$$

In fact there is a change of variables which allows (31) to be put in a simplified form. Without loss of generality we may consider the equations (31) with $i = 1$, ($i \neq 1$, is obtained from $i = 1$ by straightforward manipulation) then introduce new variables $(\chi, w, w_0) \in \mathbb{C}^{n-1} \times \mathbb{R}^n \times \mathbb{R}$ which parameterize the sphere $s(\xi)$, $(\chi = (\chi_2, \ldots, \chi_n))$

$$\begin{aligned}
k_{1_{\mathrm{R}}} &= \sum_{j=2}^{n} w_j \chi_{j_{\mathrm{R}}} - \frac{w_1}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_1}{2w^2}, \quad k_{j_{\mathrm{R}}} = -w_1 \chi_{j_{\mathrm{R}}} - \frac{w_j}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_j}{2w^2}, \\
k_{1_{\mathrm{I}}} &= \sum_{2}^{n} w_j \chi_{j_{\mathrm{I}}} + \frac{\sigma_{\mathrm{R}} w_0 w_1}{2w^2}, \quad k_{j_{\mathrm{I}}} = -w_1 \chi_{j_{\mathrm{I}}} - \frac{\sigma_{\mathrm{R}} w_0 w_j}{2w^2}, \\
\xi_1 &= \sum_{2}^{n} w_j \chi_{j_{\mathrm{R}}} - \frac{w_1}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_1}{2w^2}, \quad \xi_j = -w_1 \chi_{j_{\mathrm{R}}} - \frac{w_j}{2} - \frac{\sigma_{\mathrm{I}} w_0 w_j}{2w^2} \quad (j \geq 2).
\end{aligned} \tag{32}$$

Thus for $w_1 \neq 0$ there is a 1–1 map: $(k_R, k_I, \xi) \to (\chi, w, w_0)$ such that

$$w = \xi - k_R, \quad w = 2k_I \cdot (\xi - k_R)/\sigma_R, \tag{33a}$$

$$\frac{\partial}{\partial \bar{\chi}_j} = \mathscr{L}_{1j}, \tag{33b}$$

which for $i = 1$, $j = 2, \ldots, n$ yields

$$\frac{\partial T}{\partial \bar{\chi}_j} = N_{1j}(T)(\chi, w, w_0), \quad j = 2, \ldots, n. \tag{34}$$

Again using the generalized Cauchy formula we have

$$\mathscr{I} = T(\chi, w, w_0) - \frac{1}{\pi}\iint \frac{N_{1j}[T](\chi', w, w_0)}{\chi_j - \chi'_j}\, \mathrm{d}\chi'_{R_j}\, \mathrm{d}\chi'_{I_j} = \hat{u}(w, w_0), \tag{35}$$

where $\hat{u}(w, w_0) = \mathscr{F}(u(x, y))$ is the Fourier Transform of $u(x, y)$ with respect to w, w_0. The term $\hat{u}(w, w_0)$ is the boundary value of $T(\chi, w, w_0)$ as $\chi_j \to \infty$. This can be seen from the definition of $T(\chi, w, w_0)$ (22b) and the fact that from (32) $\chi_j \to \infty$ implies $k_j \to \infty$ and hence $\mu \to 1$. (35) leads both to admissibility criteria as well as reconstruction of $u(x, y)$. Given $T(k_R, k_I, \xi)$ one computes $\mathscr{I}$ by quadratures. We also reiterate the fact that the formula (35) assumes no homogeneous solutions to (18). We conjecture [8a] that if $\mathscr{I}$ is independent of χ and j and has suitable decay properties for large w, w_0, then T is *admissible*. The potential is recovered from

$$u(x, y) = \mathscr{F}^{-1}(\hat{u}(w, w_0)), \tag{36}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Moreover, we see that reconstruction follows purely by quadratures given $T(k_R, k_I, \xi)$ on $s(\xi) = 0$.

It turns out that the physically interesting cases of the time-dependent and time-independent Schrödinger equation in n dimensions fall out as special cases of the above result. In what follows we discuss these cases both as limits (reductions) of the above results and then briefly indicate how the formulae can be derived without recourse to any limit.

First consider the case $\sigma \to \mathrm{i}$, i.e. $\sigma_I = 1$, $\sigma_R \to 0-$ $(\sigma_R < 0)$; $\hat{k}_R, \hat{k}_I = k_I/\sigma_R$. Then $G(x, y, k_R, k_I) \to G_L(x, y, \hat{k}_R, \hat{k}_R)$ (in what follows we drop the symbol $\hat{}$),

$$G_L(x, y, k_R, k_I) = -\mathrm{i}C_n \operatorname{sgn}(y) \int \mathrm{e}^{\mathrm{i}x\cdot\xi - \mathrm{i}y(\xi^2 - 2k_R\cdot\xi)}\Theta\big(y(\xi^2 + 2(k_R + k_I)\cdot\xi)\big)\,\mathrm{d}\xi. \tag{37}$$

(37) can be directly verified, i.e.

$$\mathscr{L}G_L(x, y, k_R, k_I) = \delta(x)\delta(y), \quad \mathscr{L} = \mathrm{i}\frac{\partial}{\partial y} - \Delta + 2\mathrm{i}k_R \cdot \nabla. \tag{38}$$

and hence $\mu \to \mu_L$ where μ_L satisfies

$$\mathscr{L}\mu_L = -u\mu_L \quad \text{and} \quad \mu_L(x, y, k_R, k_I) = 1 + \tilde{G}_L(u\mu_L). \tag{39a, b}$$

Thus $G_L(x, y, k_R, k_I)$ provides a family of Green's functions parameterized by k_I which has the parameter entering via "boundary conditions" (i.e. through the integral equation (39b) since $\mathscr{L}$ depends on k_R only).

As $k_I \to \pm\infty$,

$$G_L(x, y, k_R, k_I) \to -iC_n \operatorname{sgn}(y) \int d\xi\, e^{-iy(\xi^2 - 2k_R\cdot\xi) + ix\cdot\xi}\Theta(\pm y\xi_j), \tag{40}$$

hence $G_L(x, y, k_R, k_{I_j} \to \pm\infty) \to G_L^{(\pm)j}(x, y, k_R, \tilde{k}_I)$ where $G_L^{(\pm)j}(x, y, k_R, \tilde{k}_I)$ are $\mp$ functions of k_{R_j}. Similarly $\mu_L \to_{k_{I_j} \to \pm\infty} \mu_L^{(\pm)j}(x, y, k'_R, \tilde{k}_I)$. Here $\tilde{k}_I = (k_{I_1}, \ldots, k_{I_{j-1}}, k_{I_{j+1}}, \ldots, k_n)$. Then by direct calculation (alternatively by limits):

$$\frac{\partial \mu_L}{\partial k_{I_j}} = -2iC_n \int e^{i\beta_L(x, y, k_R, k_I, \xi)} T_L(k_R, k_I, \xi)(\xi_j - k_{R_j})\delta(s_L(\xi))\mu_L(x, y, \xi, k_I)\, d\xi, \tag{41a}$$

where

$$\beta_L(x, y, k_R, k_I, \xi) = (x + 2yk_I)\cdot(\xi - k_R),$$

$$T_L(k_R, k_I, \xi) = \iint e^{-i\beta_L(x, y, k_R, k_I, \xi)}(u\mu)(x, y, k_R, k_I)\, dx\, dy; \quad s_L(\xi) = (\xi + k_I)^2 - (k_R + k_I)^2. \tag{41b}$$

The reconstruction formula for μ_L is then given by

$$\mu_L(x, y, k_R, k_I) = 1 - \frac{1}{2\pi i}\iint \left[\frac{\theta(k_{I_j} - k'_{I_j})}{k_{R_j} - k'_{R_j} - i0} + \frac{\theta(k'_{I_j} - k_{I_j})}{k_{R_j} - k'_{R_j} + i0}\right]\left(\frac{\partial \mu}{\partial k_{I_j}}\right)(x, y, k'_R, k'_I)\, dk'_{R_j}\, dk'_{I_j}, \tag{42}$$

where $k'_R \equiv (k_{R_1}, \ldots k_{R_{j-1}}, k'_{R_j}, k_{R_{j+1}}, \ldots, k_{R_n})$ and similarly for k'_I, with (41) inserted into (42). (42) can be derived directly by making use of the analytic properties of μ_L at $k_{I_j} = \pm\infty$ (or follows by limits). To show this by direct means note

$$\int_{-\infty}^{\infty} \frac{\partial \mu}{\partial k_{I_j}}\, dk_{I_j} = \mu(k_{I_j} = +\infty) - \mu(k_{I_j} = -\infty) = \mu^{(-)j}(x, y, k_R, \tilde{k}_I) - \mu^{(+)j}(x, y, k_R, \tilde{k}_I). \tag{43}$$

Thus by projection

$$\mu^{(\pm)j}(x, y, k_R, \tilde{k}_I) = 1 - P_j^{\pm}\left(\int_{-\infty}^{\infty} \frac{\partial \mu}{\partial k_{I_j}}(x, y, k_R, k_I)\, dk_{I_j}\right), \tag{44a}$$

where

$$P_j^{\pm} g(k_{R_j}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(k'_j)}{k'_j - (k_{R_j} \pm i0)}\, dk'_j \tag{44b}$$

and the usual $\pm$ projectors satisfying $(P_j^+ - P_j^-)g(k_{R_j}) = g(k_{R_j})$. Carrying out the integral in (43) from $-\infty$ to k_I and using the above boundary conditions at $k_I = -\infty$ yields

$$\mu(x, y, k_R, k_I) = 1 - P_j^- \int_{-\infty}^{\infty} \left(\frac{\partial\mu}{\partial k_I}\right) dk_I' + \int_{-\infty}^{k_I} \left(\frac{\partial\mu}{\partial k_{I_j}}\right) dk_{I_j}', \tag{45a}$$

$$= 1 - \int_{-\infty}^{k_I} P_j^- \left(\frac{\partial\mu}{\partial k_I}\right) dk_{I_j}' - \int_{k_{I_j}}^{\infty} P_j^- \left(\frac{\partial\mu}{\partial k_{I_j}}\right) dk_{I_j}' \tag{45b}$$

(with the obvious notation). (45b) is equivalent to (42). The analogue of (31) is obtained from the compatibility condition (alternatively via limits)

$$\frac{\partial^2\mu}{\partial k_{I_i}\partial k_{I_j}} = \frac{\partial^2\mu}{\partial k_{I_j}\partial k_{I_i}} \quad (i \neq j), \quad \mathscr{L}_{ij}^{L} T_L = \mathcal{N}_{ij}^{L}(T_L), \tag{46a}$$

where

$$\mathscr{L}_{ij}^{L} = \frac{i}{2}\left[(\xi_j - k_{R_i})\frac{\partial}{\partial k_{I_i}} - (\xi_i - k_{R_j})\frac{\partial}{\partial k_{I_j}}\right],$$

$$\mathcal{N}_{ij}^{L} = -C_n \int \left[(\xi_j' - k_{R_j})(\xi_i - \xi_i') - (\xi_i' - k_{R_i})(\xi_j - \xi_j')\right] \delta(s_L(\xi'))T_L(k_R, k_I, \xi')T_L(\xi', k_I, \xi)\, d\xi'. \tag{46b}$$

Defining new variables (χ, w, w_0),

$$\begin{aligned}
k_{1_R} &= \sum_2^n w_j\chi_{j_R} - \frac{w_1}{2} - \frac{w_1 w_0}{2w^2}, \quad & k_{j_R} &= -w_1\chi_{j_R} - \frac{w_j}{2} - \frac{w_j w_0}{2w^2},\\
k_{1_I} &= \sum_2^n w_j\chi_{I_j}, & k_{j_I} &= -w_1\chi_{j_I},\\
\xi_1 &= \sum_2^n w_j\chi_{j_R} + \frac{w_1}{2} - \frac{w_1 w_0}{2w^2}, & \xi_j &= -w_1\chi_{j_R} + \frac{w_j}{2} - \frac{w_0 w_j}{2w^2}, \quad j = 2, \ldots, n,
\end{aligned} \tag{47}$$

we have (taking $i = 1$ in (46))

$$\frac{\partial T_L}{\partial \chi_1} = -2i\mathcal{N}_{1j}^{L}[T_L] \tag{48}$$

and the analogue of (35)

$$T_L(\chi, w, w_0) - \frac{1}{\pi}\iint \left[\frac{\theta(\chi_1 - \chi_1')}{\chi_R - \chi_R' - i0} + \frac{\theta(\chi_1' - \chi_1)}{\chi_R - \chi_R' - i0}\right]\mathcal{N}_1[T_L](\chi', w, w_0)\, d\chi_R'\, d\chi_1' = \hat{u}(w, w_0). \tag{49}$$

Admissible data T_L are such that the LHS of (49) will be independent of χ, and j and have appropriate decay in w, w_0. (49) can be derived directly using the fact that $\lim_{\chi_I \to \pm\infty}(T_L - \hat{u})$ is, as a function of χ_R, the boundary value of an analytic function of χ_R in the lower $(\chi_R \to +\infty)$/upper $(\chi_R \to -\infty)$ half-plane and tending to zero as $\chi_R \to \infty$. The argument is identical to that of reconstructing μ_L above (i.e. (43)–(45)).

Next we give the results for the stationary case: i.e. $u(x, y) = u(x)$. The methods to obtain these formulae follow from those of μ_L by reduction or alternatively can be verified directly using the same techniques as those described above and hence will be omitted, apart from illuminating comments. The "stationary" eigenfunction $\mu_s(x, k_R, k_I)$ satisfies

$$\mu_s(x, k_R, k_I) = 1 + \tilde{G}_s(u\mu_s), \tag{50}$$

where the Green's function is given by

$$G_s(x, k_R, k_I) = -C_n \int \left(\frac{\theta(k_I \cdot \xi)}{\xi^2 + 2\xi \cdot k_R - i0} + \frac{\theta(-k_I \cdot \xi)}{\xi^2 + 2\xi \cdot k_R + i0} \right) e^{i\xi \cdot x} d\xi. \tag{51}$$

Hereafter we assume that (50) has no homogeneous solutions. By direct calculations it can be verified that G_s satisfies

$$\mathscr{L}_s G_s(x, k_R, k_I) = \delta(x), \tag{52a}$$

$$\mathscr{L}_s = \Delta + 2i k_R \cdot \nabla \tag{52b}$$

and μ_s satisfies

$$\mathscr{L}_s \mu_s = u\mu_s \tag{53a}$$

or, if $\psi_s(x, k_R, k_I) = \mu_s e^{ik_R \cdot x}$, ψ_s satisfies

$$(\Delta + k_R^2 - u(x))\psi_s = 0. \tag{53b}$$

G_s is obtained from G_L by

$$G_s(x, k_R, k_I) = \int_{-\infty}^{\infty} G_L(x, y, k_R, k_I) \, dy, \tag{54}$$

where the identity

$$\frac{\theta(x+y)}{x - i0} + \frac{\theta(-x-y)}{x + i0} = \frac{\theta(y)}{x - i0} + \frac{\theta(-y)}{x + i0}$$

is useful. Indeed the Green's function $G_s(x, k_R, k_I)$ turns out to be the same as that of Faddeev [10]!

The analogue to (23) is now

$$\frac{\partial \mu_s}{\partial k_I} = -2i\pi C_n \int e^{i(x+2yk_I)\cdot(\xi - k_R)} T_s(k_R, k_I, \xi)(\xi_j - k_R)\delta(\xi^2 - k_R^2)\delta(k_I \cdot (\xi - k_R))\mu(x, \xi, k_I) \, d\xi. \tag{55a}$$

where

$$T_s(k_R, k_I, \xi) = \int e^{-ix\cdot(\xi - k_R)} u(x)\mu_s(x, k_R, k_I)\,dx. \tag{55b}$$

It should be remarked that the reduction of T_L to T_s obeys

$$T_L(k_{R,I}, \xi) = \pi\delta(k_I\cdot(\xi - k_R))T_s(k_R, k_I, \xi). \tag{55c}$$

The reconstruction equation for $\mu_s(x, k_R, k_I)$ follows directly (noting that as $k_{I_j} \to \pm\infty$ μ_s is a $\mp$ function of k_{R_j}):

$$\mu_s(x, y, k_R, k_I) = 1 - \frac{1}{2\pi i}\int\left[\frac{\theta(k_{I_j} - k'_{I_j})}{k_{R_j} - k'_{R_j} - i0} + \frac{\theta(k'_{I_j} - k_{I_j})}{k_{R_j} - k'_{R_j} + i0}\right]\left(\frac{\partial\mu_s}{\partial k_{I_j}}\right)(x, y, k'_R, k'_I)\,dk'_R\,dk'_I. \tag{56}$$

using (55).

Taking the restriction $w_0 = k_I\cdot(\xi - k_R) = 0$ into account then the compatibility $\partial^2\mu_s/\partial k_{I_i}\partial k_{I_j} = \partial^2\mu_s/\partial k_{I_j}\partial k_{I_i}$ $(i \neq j)$ yields

$$\mathscr{L}^s_{ij}T_s = \pi N^s_{ij}(T_s) \quad \text{on } w_0 = 0. \tag{57a}$$

where

$$\mathscr{L}^s_{ij} = \frac{i}{2}\left((\xi_j - k_{R_j})\frac{\partial}{\partial k_{I_i}} - (\xi_i - k_{R_i})\frac{\partial}{\partial k_{I_j}}\right). \tag{57b}$$

$$N^s_{ij}(T_s) = -C_n\int\left[(\xi'_j - k_{R_j})(\xi_i - \xi'_i) - (\xi'_i - k_{R_i})(\xi_j - \xi'_j)\right]\delta(\xi'^2 - k_R^2)$$
$$\times\,\delta(k_I\cdot(\xi' - k_R))T_s(k_R, k_I, \xi')T_s(\xi', k_I, \xi)\,d\xi'. \tag{57c}$$

The change of variables $(w_0 = 0)$

$$k_{1_R} = \sum_2^n w_j\chi_{R_j} - \frac{w_1}{2}, \quad k_{j_R} = -w_1\chi_{j_R} - \frac{w_j}{2}, \quad k_{1_I} = \sum_2^n w_j\chi_{I_j},$$
$$k_{j_I} = -w_1\chi_{j_I}, \quad \xi_1 = \sum_2^n w_j\chi_{j_R} + \frac{w_1}{2}, \quad \xi_j = -w\chi_{j_R}, \quad j = 2, \ldots, n, \tag{58}$$

gives a transformation $(k_R, k_I, \xi) \to (\chi, w)$ from $3n - 2$ variables to $3n - 2$ variables (note we have the restrictions $\xi^2 = k_R^2$ and $w_0 = k_I\cdot(\xi - k_R) = 0$ incorporated into this transformation). Thus (57)–(58) implies (taking $i = 1$ in (57))

$$\frac{\partial T_s}{\partial\chi_{1_j}} = -2i\pi N_{1j}(T_s) \tag{59}$$

and then by integration

$$T_s(\chi,w) - \iint \left[\frac{\theta(\chi_{I_j} - \chi'_{I_j})}{\chi_{R_j} - \chi'_{R_j} - i0} + \frac{\theta(\chi'_{I_j} - \chi_{I_j})}{\chi_{R_j} - \chi'_{R_j} + i0} \right] N_{1j}[T_s](\chi',w)\, d\chi'_R\, d\chi'_I = \hat{u}(w), \tag{60}$$

where $\hat{u}(w)$ is the Fourier Transform of $u(x)$ with respect to w. (60) plays the role of characterizing suitable data $T_s(x,w)$ as well as reconstructing the potential in analogy with (35). Namely, the LHS of (60) must be independent of χ and j and have appropriate decay in w. Again we note that $\lim_{\chi_{I_j} \to \pm\infty}(T_s - \hat{u}(w))$ is as a function of χ_{R_j} the boundary value of an analytic function in the lower $(+\infty)$/upper $(-\infty)$ half-plane and vanishing at $\chi_{R_j} = \infty$. This allows (60) to be obtained directly by using the same ideas discussed earlier.

Next we show how the inverse data described earlier i.e. $T(k_R,k_I,\xi)$, $T_L(k_R,k_I,\xi)$, $T_s(k_R,k_I,\xi)$ for the general case and limit/reduction cases can be related to scattering data. In the limit (L)/reduction (S) case scattering theory has a clear physical meaning. In the general case we shall define formal scattering and show how time dependent/independent physical scattering can be recovered as special cases. Naturally one can derive these latter results ((L),(S)) directly. Since such an analysis is essentially identical to the general case and will be omitted. Also remark that such formulae for the time independent case was originally derived by Faddeev [10].

We begin by defining a "left-Volterra" operator in terms of a Green's function.

$$G_v(x,y,k) = \frac{C_n}{\sigma}\Theta(y)\int e^{ix\cdot\xi + (\xi^2 + 2k\cdot\xi)y/\sigma}\, d\xi, \tag{61}$$

where $k = k_R + ik_I$ and we will require $\sigma_R < 0$ for convergence. Then for functions $u(x,y)$ of compact-support in both x and y (a much wider function class is allowed in the limit/reduction cases):

$$\mu_v(x,y,k) = 1 + \bar{G}_v(u\mu_v)(x,y,k). \tag{62}$$

The scattering function is defined by the limit $y \to \infty$ of (62) (as $y \to -\infty$, $\mu_v \to 1$):

$$\mu_v(x,y,k) = 1 + \int d\xi\, e^{ix\cdot\xi + (\xi^2 - 2k\cdot\xi)y/\sigma}\tilde{S}(k_R,k_I,\xi). \tag{63a}$$

$$\tilde{S}(k_R,k_I,\xi) = \frac{C_n}{\sigma}\iint e^{-i\xi\cdot x - (\xi^2 + 2k\cdot\xi)y/\sigma}(u\mu_v)(x,y,k)\, dx\, dy, \tag{63b}$$

or, by changing variables $\xi \to \xi - k_R$,

$$\hat{S}(k_R,k_I,\xi) = \frac{C_n}{\sigma}\iint e^{-\hat{\beta}(x,y,k_R,k_I,\xi)}(u\mu_v)(x,y,k)\, dx\, dy, \tag{63c}$$

where

$$\hat{\beta}(x,y,k_R,k_I,\xi) = ix\cdot(\xi - k_R) + (\xi^2 - k_R^2 - 2ik_I\cdot(\xi - k_R))y/\sigma. \tag{63d}$$

The eigenfunctions μ and μ_v are related as follows:

$$(\mu-\mu_v)=\bar{G}(u\mu)-\bar{G}_v(u\mu_v)=\bar{G}_v(u(\mu-\mu_v))+(\bar{G}-\bar{G}_v)(u\mu). \tag{64}$$

Using (20) and (61) we have

$$(G-G_v)(x,y,k_R,k_I)=\frac{-C_n}{\sigma}\int e^{\hat{\beta}(x,y,k_R,k_I,\xi)}\Theta(-s(\xi))\,d\xi, \tag{65a}$$

hence

$$(\bar{G}-\bar{G}_v)(u\mu)=\int e^{\hat{\beta}(x,y,k_R,k_I,\xi)}\Theta(-s(\xi))\hat{T}(k_R,k_I,\xi)\,d\xi, \tag{65b}$$

where

$$\hat{T}(k_R,k_I,\xi)=-\frac{C_n}{\sigma}\iint e^{-\hat{\beta}(x,y,k_R,k_I,\xi)}(u\mu)(x,y,k_R,k_I)\,dx\,dy. \tag{65c}$$

Note when $s(\xi)=(\xi-k_R\sigma_I/\sigma_R)^2-(k_I-k_I\sigma_I/\sigma_R)^2=0$. Then $\hat{\beta}(x,y,k_R,k_I,\xi)=\beta(x,y,k_R,k_I,\xi)$ and $\hat{T}(k_R,k_I,\xi)=T(k_R,k_I,\xi)$ (see 22).

Then employing the symmetry condition on G_v,

$$e^{-\hat{\beta}(x,y,k_R,k_I,\xi)}G_v(x,y,k_R,k_I)=G_v(x,y,\xi,k_I) \tag{66}$$

(which is verified directly), we have from (64)–(66)

$$(\mu-\mu_v)(x,y,k_R,k_I)=\int e^{\hat{\beta}(x,y,k_R,k_I,\xi')}\Theta(-s(\xi'))\hat{T}(k_R,k_I\xi')\mu_v(x,y,\xi',k_I)\,d\xi'. \tag{67}$$

Multiply (67) by $(C_n/\sigma)u(x,y)e^{-\hat{\beta}(x,y,k_R,k_I,\xi)}$ and take $\iint dx\,dy$. Then we find with the definitions of $\hat{T}$, $\hat{S}$ (63, 65)

$$\hat{T}(k_R,k_I,\xi)+\hat{S}(k_R,k_I,\xi)-\int\Theta(-s(\xi'))\hat{T}(k_R,k_I,\xi')\hat{S}(\xi',k_I,\xi)\,d\xi'=0. \tag{68}$$

(68) yields $\hat{T}(k_R,k_I,\xi)$ given the "scattering data" $\hat{S}(k_R,k_I,\xi)$.

The limit/reduction cases now follow immediately. For $\sigma\to i$, $\sigma_R<0$

$$\hat{\beta}\to\hat{\beta}_L(x,y,k_R,\xi)=i(x\cdot(\xi-k_R)-(\xi^2-k_R^2)y), \tag{69a}$$

$$\hat{S}\to\hat{S}_L(k_R,\xi)=-iC_n\iint e^{-\hat{\beta}_L(x,y,k_R,\xi)}(u\mu_v^L)(x,y,k_R)\,dx\,dy. \tag{69b}$$

Note in (69a) $\hat{\beta}_L(x,y,k_R,\xi)=\beta_L(x,y,k_R,k_I,\xi)=i(x+2yk_I)\cdot(\xi-k_R)$ on the "shell" $s_L(\xi)=(\xi+k_I)^2-(k_R+k_I)^2=0$. In (69b) μ_v^L is defined by

$$\mu_v^L(x,y,k_R)=1+\bar{G}_v^L(u\mu_v^L)(x,y,k_R). \tag{69c}$$

where

$$G_\nu^{\mathrm{L}}(x, y, k_{\mathrm{R}}) = -\mathrm{i}C_n\Theta(y)\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\cdot\xi - \mathrm{i}(\xi^2 - 2k_{\mathrm{R}}\cdot\xi)y}\,\mathrm{d}\xi. \tag{69d}$$

μ_ν^{L} is not a function of k_{I} hence neither is $\hat{S} = \hat{S}(k_{\mathrm{R}}, \xi)$. Similarly:

$$\hat{T} \to \hat{T}_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi) = \mathrm{i}C_n \iint \mathrm{e}^{-\hat{\beta}_{\mathrm{L}}(x, y, k_{\mathrm{R}}, \xi)}(u\mu)(x, y, k_{\mathrm{R}}, k_{\mathrm{I}})\,\mathrm{d}x\,\mathrm{d}y. \tag{69e}$$

Thus we have the scattering relationship

$$\hat{T}_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi) + \hat{S}_{\mathrm{L}}(k_{\mathrm{R}}, \xi) + \int \Theta(-s_{\mathrm{L}}(\xi'))\hat{T}_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi')\hat{S}_{\mathrm{L}}(\xi', \xi)\,\mathrm{d}\xi' = 0. \tag{70}$$

Again given the scattering function $\hat{S}_{\mathrm{L}}(k_{\mathrm{R}}, \xi)$ in principle we can obtain $\hat{T}_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ from (70) and this equals $T_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ (see (41)) on $s_{\mathrm{L}}(\xi) = 0$. For the time independent (reduction) problem we make the observation

$$G_\nu^{\mathrm{s}}(x, k_{\mathrm{R}}) = \int_{-\infty}^{\infty} G_\nu^{\mathrm{L}}(x, y, k_{\mathrm{R}})\,\mathrm{d}y = -C_n\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x\cdot\xi}}{\xi^2 - 2k_{\mathrm{R}}\cdot\xi - \mathrm{i}0}\,\mathrm{d}\xi \equiv G_{-}(x, k_{\mathrm{R}}). \tag{71a}$$

Namely $G_\nu^{\mathrm{s}}(x, k$ is identical to the standard outgoing Green's function (which is also analytic in the upper half $s = |k_{\mathrm{R}}|$ plane. Thus

$$\mu_\nu^{\mathrm{s}}(x, k_{\mathrm{R}}) = \mu_{-}(x, k_{\mathrm{R}}). \tag{71b}$$

Using $\int \mathrm{e}^{-\mathrm{i}y(\xi^2 - k_{\mathrm{R}})}\,\mathrm{d}y = 2\pi\delta(\xi^2 - k_{\mathrm{R}}^2)$, we have the identifications when $u(x, y) = u(x)$

$$\hat{S}_{\mathrm{L}}(k_{\mathrm{R}}, \xi) = -C_{n-1}\delta(\xi^2 - k_{\mathrm{R}}^2)A(k_{\mathrm{R}}, \xi), \tag{71c}$$

$$\hat{T}_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi) = \mathrm{i}C_{n-1}\delta(\xi^2 - k_{\mathrm{R}}^2)T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi), \tag{71d}$$

where

$$A(k_{\mathrm{R}}, \xi) = \int \mathrm{e}^{-\mathrm{i}x\cdot(\xi - k_{\mathrm{R}})}(u\mu_{-})(x, k_{\mathrm{R}})\,\mathrm{d}x \tag{71e}$$

and $T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ is defined by (55b). Then (70) reduces to

$$T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi) - A(k_{\mathrm{R}}, \xi) - \mathrm{i}C_{n-1}\int \theta(k_{\mathrm{I}}\cdot(k_{\mathrm{R}} - \xi'))\delta(\xi'^2 - k_{\mathrm{R}}^2)T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi')A(\xi', \xi)\,\mathrm{d}\xi' = 0, \tag{72}$$

on $\xi^2 = k_{\mathrm{R}}^2$. (72) was obtained by Faddeev [10] in his study of the time-dependent Schrödinger problem and serves to relate the physical outgoing scattering amplitude to the "inverse data" $T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ (also called the nonphysical scattering amplitude). We reiterate the fact that the derivation above could be carried out directly on the time-dependent/independent Schrödinger operator without any recourse to "generalized scattering" as we have introduced it here.

3. Concluding remarks

(i) $T(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$ (and $T_{\mathrm{L}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi), T_{\mathrm{s}}(k_{\mathrm{R}}, k_{\mathrm{I}}, \xi)$) satisfy a quadratically nonlinear differential-integral equation when $n > 1$; i.e. (31) (and (46), (57)). The fundamental feature of this equation is that it leads to characterization/admissibility criteria for the inverse data. However at the same time it precludes the existence of a simple time evolution of the data i.e. $T(\cdot, t) \neq T(\cdot, 0)e^{i\omega t}$. Such simple flows are associated with the KdV, KP etc. equations. Hence this result provides still another explanation for why local nonlinear evolution equations have not been associated with the multidimensional scattering problem (14).

(ii) Eqs. (35) and its limit/reduction cases (49), (60) provide characterization/admissibility criteria for the inverse data and a reconstruction formulae for the potential in the same formula. Even for the classical problem of the time-independent Schrödinger operator (cf. [10, 11] our eqs. (60) yield some novel information: it shows that Faddeev's characterization (with which it is essentially equivalent cf. [8c]) naturally arises as an integral equation for T, a somewhat more convenient condition to verify than his analyticity requirement; it also shows that once T is known the potential can be found purely by quadratures. The scattering data are related to the inverse data via the formulae (68), (70), (72). For (70), (72) the scattering amplitudes are physically relevant and, in principle, measurable. It is an open and important problem regarding how one could measure the scattering amplitude and at the same time ensure that the inverse data resulting from (70), (72) will still be admissible even when small errors are present. Namely, how can one adjust errors in data in order to ensure admissibility.

(iii) Although here we have discussed the analysis for the generalized Schrödinger scattering problem, the algorithm also works other operators in a straightforward way. In [8b] the scattering problem (see (v) in the introduction):

(iv)

$$v_y + \sigma \sum_{l=1}^{n} J_l v_{x_l} = q(x, y) v,$$

with $\sigma = \sigma_{\mathrm{R}} + i\sigma_{\mathrm{I}}$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, q an $N \times N$ matrix, and $J_l = \mathrm{diag}(J_l^1, \ldots, J_l^N)$. Again results analogous to (31) follow; i.e. the scattering data satisfies a nonlinear equation. On the other hand, (iv) is one of the few operators that has a compatible time evolution operator and hence a Lax pair describing a nonlinear evolution equation in multidimensions: the so-called N-wave interaction equation. But the N-wave equations can hold only if certain restrictions are put on J_l^i: namely that the vectors $J^i = (J_1^i, J_2^i, \ldots, J_n^i)$ are all colinear. In this case the coefficient of the nonlinear term in the equation for T vanishes – i.e. the analogy to (31) is now *purely linear* and it allows a simple flow in time and the N-wave equation follows and is solvable by IST. Nevertheless, despite the fact that the N-wave equation is formally multidimensional, new variables may be introduced to reduce the problem to two spatial dimensions. The colinearity of the vectors J^i allows a reduction to three spatial dimensions [8b] and the introduction of appropriate characteristic coordinates further reduces the N-wave equation down to two spatial dimensions [14]. Apart from this special case the analysis suggests there will not be other local nonlinear evolution equations compatible with (iv) (see also [15]).

(v) Prototype operators such as those discussed in this paper provide a convenient testing ground for the development of scattering (and also IST) theories which one hopes can also be applied to other physically interesting models.

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NOTE ON THE INVERSE PROBLEM FOR A CLASS OF FIRST ORDER MULTIDIMENSIONAL SYSTEMS

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ABSTRACT. The inverse problem for a multidimensional system of first order differential equations is considered. The $\bar{\partial}$ methodology is employed and integral equations are developed for which the potential may be reconstructed.

In recent years there has been substantial interest in the study of: (a) inverse scattering problems for appropriately decaying potentials (i.e. given suitable scattering data reconstruct the potential $q(x)$); (b) the initial value problem of certain physically important nonlinear evolution equations (i.e. given $q(x,0)$ find $q(x,t)$). In this note we shall consider the inverse problem associated with

$$\psi_{x_0} + i\sum_{\ell=1}^{n} J_\ell \psi_{x_\ell} = q\psi, \tag{1}$$

where $q(x_0,x)$ is an $N \times N$ matrix-valued off-diagonal function in $\mathbb{R}^{n+1}$ and J_ℓ are constant real diagonal $N \times N$ matrices (we denote the diagonal entries of J_ℓ by $J_\ell^1,\ldots,J_\ell^N$). We note that the methods presented here can be easily extended to the system $\psi_{x_0} + \sigma\sum_{\ell=1}^{n} J_\ell\psi_{x_\ell} = q\psi$, $\sigma = \sigma_R + i\sigma_I$, which as $\sigma_I \to 0$ becomes the linear eigenvalue problem associated with the so called N wave-interaction equation in $n+1$ spatial dimensions (see [1]). Associated with (1) is a nonlinear evolution equation (a complexified form of the N-wave equation) which is in a sense illposed. Nevertheless (1) provides a natural scattering system to study with the methods at our disposal.

Using the transformation $\psi(x_0,x,k)=\mu(x_0,x,k)\exp(i\Sigma k_\ell(x_\ell - ix_0 J_\ell))$; $k \in \mathbb{C}^n$, we may alternatively consider the system

$$\mu_{x_0} + \sum_{\ell=1}^{n} (iJ_\ell\mu_{x_\ell} - k_\ell[J_\ell,\mu]) = q\mu. \tag{2}$$

Equations (1), (2) are natural extensions of well known problems:

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i) In one spatial dimension, i.e. $n=0$, (1) would correspond to

$$\psi_{x_0} + ikJ\psi = q(x_0)\psi, \quad -\infty < x_0 < \infty. \tag{3}$$

The transformation $\psi(x_0,k) = \mu(x_0,k)e^{-ikJx_0}$, $k \in \mathbb{C}$ leads to the system of differential equations

$$\mu_{x_0} + ik[J,\mu] = q\mu. \tag{4}$$

The function $\mu(x_0,k)$ has desirable analytic properties in k, provided that q is in an appropriate space. Utilization of these analytic properties leads to the formulation of a Riemann-Hilbert (RH) problem for the solution of the inverse problem associated with (4). The 2×2 case has been studied in [2], [3]; it can be used to solve the initial value problem of the nonlinear Schrödinger, Sine-Gordon, and modified Korteweg-deVries equation. The 3×3 case was studied in [4]; it can be used to solve the initial value problem of the 3 wave interaction (a review of the above work appears in [5]). Recently the $N \times N$ case was studied by a number of authors and in a completely rigorous manner by Beals and Coifman [6,7].

ii) In two spatial dimensions (i.e. $n=1$) equations (1) and (2) were studied in [8]. The inverse problem was formulated and formally solved in terms of a DBAR ($\bar{\partial}$) problem (a $\bar{\partial}$ problem generalizes the notion of a RH problem). The 2×2 case of this inverse problem was used to solve the initial value problem of certain nonlinear evolution equations in two spatial and one temporal dimension: the Modified Kadomtsev-Petviashvili II (MKPII), and Davey-Stewartson II (DSII) equation. The hyperbolic analogs of (1), (2) (i.e. $J_\ell \to iJ_\ell$)) in two spatial dimensions (i.e. $n=1$) was studied in [9]. The inverse problem in this case was adequately treated via a RH problem; it was used to solve the initial value problem of the N wave interaction, MKPI and DSI.

The solution of the inverse problem associated with (2) has two aspects: (a) develop a formalism such that given appropriate inverse data $T^{ij}(k,\lambda)$ one may reconstruct the potential $q(x_0,x)$. (b) It turns out that $T^{ij}(k,\lambda)$ depends on $3n-1$ parameters while the potential depends only on $n+1$. Thus one needs a characterization equation that restricts the scattering data. In this note we only consider (a) above by extending the method of [8,10], question (b) is considered in [1].

In component form equation (4) is written as:

$$(L\mu)^{ij} = \mu^{ij}_{x_0} + \sum_{\ell=1}^{n} iJ^i_\ell \mu^{ij}_{x_\ell} - k_\ell(J^i_\ell - J^j_\ell)\mu^{ij} = (q\mu)^{ij} . \tag{5}$$

The specific eigenfunctions we shall work with are defined by the integral equations:

$$\mu = I + \tilde{G}(q\mu) \tag{6a}$$

or

$$\mu^{ij}(x_0,x,k)=\delta_{ij}+\int_{\mathbb{R}^{n+1}} G^{ij}(x_0-y_0,x-y,k)(q(y_0,y)\mu(y_0,y,k))^{ij}dy_0dy \quad (6b)$$

where δ_{ij} is the usual Kronecker delta function. The Green's function satisfies: $(LG)^{ij}=\delta(x_0-y_0)\delta(x-y)$ and is given by:

$$G^{ij}(x_0,y_0,k)=\frac{\operatorname{sgn}(J_1^i)}{2\pi i(x_1-iJ_1^i\,x_0)}\,e^{i\alpha^{ij}(x_0,x,k)}\prod_{\ell=2}^{n}\delta\Big(x_\ell-\frac{J_\ell^i}{J_1^i}x_1\Big),$$

where

$$\alpha^{ij}(x_0,x,k)=\sum_{\ell=1}^{n}(J_\ell^i-J_\ell^j)\Big(x_0k_{\ell_I}-\frac{x_\ell k_{\ell_R}}{J_\ell^i}\Big)\,. \quad (7)$$

(7) is obtained by looking for a Fourier representation of the Green's function whereby one finds:

$$G^{ij}(x_0,x,k)=\frac{-i}{(2\pi)^{n+1}}\int\frac{e^{i(x_0\xi_0+x\cdot\xi)}}{\xi_0+i\sum_{\ell=1}^{n}[(J_\ell^i\xi_\ell+k_\ell(J_\ell^i-J_\ell^j)]}\,d\xi_0d\xi. \quad (8)$$

(7) is then calculated by using:

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi+a+ib}\,d\xi=2\pi i\,\operatorname{sgn}(x)e^{-i(a+ib)x}\,\theta(-xb)$$

$$\int_{c\xi<A}e^{(c+id)\xi}d\xi=\frac{\operatorname{sgn}(c)}{c+id}\,e^{(c+id)A/c},\quad c\neq 0$$

where the heaviside function is defined by: $\theta(x)=\{1,\ x>0;\ x,\ x<0\}$. We next show that $\partial\mu/\partial\bar{k}_p$ (where $\partial/\partial\bar{k}_p=\frac{1}{2}(\partial/\partial k_{pR}+i\,\partial/\partial k_{pI})$) can be expressed in terms of μ. From (6a) we have

$$\frac{\partial\mu}{\partial\bar{k}_p}=\frac{\partial\tilde{G}}{\partial\bar{k}_p}(q\mu)+\tilde{G}\Big(q\frac{\partial\mu}{\partial\bar{k}_p}\Big) \quad (9)$$

and by direct calculation

$$\frac{\partial G^{ij}}{\partial\bar{k}_p}=i\Big(\frac{J_p^i-J_p^j}{2}\Big)\Big(x_0+i\,\frac{x_p}{J_p^i}\Big)G^{ij} \quad (10a)$$

$$=-\frac{1}{2}\,\frac{(J_p^i-J_p^j)}{(2\pi)^n}\int_{\mathbb{R}^n}\delta\Big(\sum_{\ell=1}^{n}J_\ell^i\,\lambda_\ell\Big)e^{i\beta^{ij}(x_0,x,k,\lambda)}d\lambda \quad (10b)$$

where

$$\beta^{ij}(x_0,x,k,\lambda)=\alpha^{ij}(x_0,x,k)+\sum_{\ell=1}^{n}x_\ell\lambda_\ell\,.$$

Defining the scattering data:

$$T^{ij}(k,\lambda) = \int_{\mathbb{R}^{n+1}} e^{-i\beta^{ij}(y_0,y,k,\lambda)}(q\mu)^{ij}(y_0,y,k)dy_0dy \tag{11}$$

(9) may be written in the form:

$$\left(\frac{\partial\mu}{\partial\bar{k}_p}\right)^{ij}(x_0,x,k) = -\frac{1}{2}\frac{(J_p^i - J_p^j)}{(2\pi)^n}\int_{\mathbb{R}^n} \delta\left(\sum_{\ell=1}^{n} J_\ell^i\lambda_\ell\right)e^{i\beta^{ij}(x_0,x,k,\lambda)}T^{ij}(k,\lambda)d\lambda$$

$$+ \int_{\mathbb{R}^{n+1}} G^{ij}(x_0-y_0,x-y,k)\left(q\frac{\partial\mu}{\partial\bar{k}_p}\right)^{ij}(y_0,y,k)dy_0dy. \tag{12}$$

In order to express $\partial\mu/\partial\bar{k}_p$ in terms of μ we decompose $\partial\mu/\partial\bar{k}_p$ into fundamental matrices $M_{\nu\nu'}(x_0,x,k,\lambda)$ on $\sum_{\ell=1}^{n} J_\ell^\nu\lambda_\ell = 0$:

$$M_{\nu\nu'}(x_0,x,k,\lambda) = e^{i\beta^{\nu\nu'}(x_0,x,k,\lambda)}E_{\nu\nu'} + \tilde{G}(qM_{\nu\nu'})(x_0,x,k) \tag{13}$$

where the elementary matrix $E_{\mu\nu}$ has components:

$$(E_{\nu\nu'})^{ij} = \begin{cases} 1 & \nu=i,\ \nu'=j \\ 0 & \text{otherwise} \end{cases}.$$

Hence once we have $M_{\nu\nu'}$ then we have $\partial\mu/\partial\bar{k}_p$ via:

$$\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k) = \sum_{\nu,\nu'=1}^{n}\left(-\frac{1}{2}\right)\frac{(J_p^\nu-J_p^{\nu'})}{(2\pi)^n}\int_{\mathbb{R}^n}\delta\left(\sum_{\ell=1}^{n} J_\ell^\nu\lambda_\ell\right)e^{i\beta^{\nu\nu'}(x_0,x,k,\lambda)}T^{\nu\nu'}(k,\lambda)E_{\nu\nu'}d\lambda$$

$$+ \tilde{G}\left(q\frac{\partial\mu}{\partial\bar{k}_p}\right)(x_0,x,k) \tag{14a}$$

and hence

$$\frac{\partial\mu}{\partial\bar{k}_p} = \sum_{\nu,\nu'=1}^{n} -\left(\frac{1}{2}\right)\frac{(J_p^\nu-J_p^{\nu'})}{(2\pi)^n}\int_{\mathbb{R}^n}\delta\left(\sum_1^n J_\ell^\nu\lambda_\ell\right)T^{\nu\nu'}(k,\lambda)M_{\nu\nu'}(x_0,x,k,\lambda)d\lambda. \tag{14b}$$

From (14) it is clear that the only nontrivial combinations come from columns of $(\partial\mu/\partial\bar{k}_p)^{ij}$ such that $j=\nu'$. Letting $(n_{\nu j})^{rj} = (M_{\nu j})^{rj}e^{-i\beta^{\nu j}}$, we have

$$(n_{\nu j})^{rj}(x_0,x,k) = \delta_{r\nu} + \int_{\mathbb{R}^{n+1}} e^{-i\beta^{\nu j}(x_0,x,k,\lambda)} \times$$

$$\times G^{rj}(x_0-y_0,x-y,k)e^{i\beta^{\nu j}(y_0,y,k,\lambda)}(qn_{\nu j})^{rj}(y_0,y,k)dy_0dy \tag{15}$$

The fact that the Greens function admits the following symmetry condition:

$$e^{-i\beta^{\nu j}(x_0,x,k,\lambda)}G^{rj}(x_0,x,k) = G^{r\nu}(x_0,x,\hat{k}^{\nu j}(k,\lambda)),\ \text{on}\ \sum_{\ell=1}^{n} J_\ell^\nu\lambda_\ell=0, \tag{16a}$$

where

$$\hat{k}^{\nu n}(k,\lambda)=\left(\frac{J_\ell^j}{J_\ell^\nu}k_{\ell R}+\lambda_\ell, k_{\ell I}\right)_{\ell=1,\ldots,n}, \text{ with } \lambda_\ell \text{ satisfying } \sum_{\ell=1}^{n} J_\ell^\nu \lambda_\ell = 0. \quad (16b)$$

(16b) immediately gives:

$$(n_{\nu j})^{rj}(x_0,x,k) = \mu^{r\nu}(x_0,x,\hat{k}^{\nu j}(k,\lambda))$$

whereupon from (14) we have

$$\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k) = \sum_{\nu,\nu'}\left(-\frac{1}{2}\right)\frac{(J_p^\nu - J_p^{\nu'})}{(2\pi)^n}\int_{\mathbb{R}^n}\delta\left(\sum_{\ell=1}^{n} J_\ell^\nu\lambda_\ell\right)T^{\nu\nu'}(k,\lambda)e^{iB^{\nu\nu'}(x_0,x,k,\lambda)} \times$$

$$\times\ \mu(x_0,x,\hat{k}^{\nu\nu'}(k,\lambda))E_{\nu\nu'}\, d\lambda. \quad (17)$$

It should be noted that (16a) is suggested by the transformation between bounded eigenfunctions of (3). To see this explicitly, note that if ψ is a solution of (3) then so is $\psi E_{\nu\nu'}$, and therefore the function $\nu(x_0,x,k) = \psi E_{\nu\nu'}\exp(i\Sigma k_\ell(x_\ell - ix_0J_\ell))$ satisfies (4). But since the function $\mu(x_0,x,h)\exp(i\Sigma h_\ell(x_\ell - ix_0J_\ell))$ also satisfies (4) we have the transformation law:

$$\nu(x_0,x,k) = \mu(x_0,x,h)e^{i\sum_{\ell=1}^{n} h_\ell(x_\ell - ix_0J_\ell)} E_{\nu\nu'}e^{-i\sum_{\ell=1}^{n} k_\ell(x_\ell - ix_0J_\ell)} \quad (18)$$

For boundedness we require:

$$\mathrm{Re}\{i\sum_{\ell=1}^{n}[(h_\ell - k_\ell)x_\ell - (h_\ell J_\ell^\nu - k_\ell J_\ell^{\nu'})ix_0]\} = 0. \quad (19)$$

hence:

$$h = \hat{k}^{\nu\nu'}(k,\lambda) = \left(\frac{J_\ell^{\nu'}}{J_\ell^\nu}k_{\ell R} + \lambda_\ell,\ k_{\ell I}\right)_{\ell=1,\ldots,n} \quad (20a)$$

for any λ_ℓ on

$$\sum_{\ell=1}^{n}\lambda_\ell J_\ell^\nu = 0. \quad (20b)$$

Finally the reconstruction is effected by inverting $\bar{\partial}\mu$ one variable at a time:

$$\mu(x_0,x,k) = I + \frac{1}{\pi}\iint_{\mathbb{R}^2}\frac{\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k_1,\ldots k_p',\ldots k_n)}{k_p - k_p'}\,dk_{p_R}'\,dk_{p_I}' \quad (21)$$

and using (17) to obtain a linear integral equation for μ. Asymptotically, as $k_p \to \infty$, (21) yields

$$\mu \sim I + \frac{1}{\pi k_p}\iint\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k_1,\ldots,k_p',\ldots,k_n)dk_{p_R}'\,dk_{p_I}'. \quad (22)$$

On the other hand substituting the asymptotic expansion

$$\mu \sim I + \frac{1}{k_p}\mu^{(1)} + \ldots$$

into (4) gives the relation:

$$q = -[J_p,\mu^{(1)}]$$

from which we have the formula:

$$q(x_0,x) = \frac{1}{\pi}\left[\iint \frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k_1,\ldots,k_p',\ldots,k_n)\,dk'_{p_R}\,dk'_{p_I}, J_p\right], \tag{23}$$

with (17) used in (23).

Formulae (17), (21), (23) can be used for the reconstruction of $q(x_0,x)$. At this point one needs to show that: (a) $q(x_0,x)$ given by (23) is independent of $k_1,\ldots,k_{p-1}$, $k_{p+1},\ldots,k_n$; (b) the same $q(x_0,x)$ is found regardless of which inversion formula is used ($p=1,\ldots,n$); (c) there exists a restriction on the scattering data $T^{ij}(k,\lambda)$, which has $3n-1$ parameters whereas $q(x_0,x)$ has only $n+1$. It can be easily shown that (i), (ii) are equivalent. Furthermore there exists a characterization equation restricting the scattering data T^{ij}, this equation is given in [1].

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Nonlinear evolution equations associated with a Riemann–Hilbert scattering problem

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In an earlier paper nonlinear evolution equations associated with a Riemann–Hilbert scattering problem, which reduces, in an appropriate limit, to the Zakharov–Shabat–AKNS scattering problem, were considered. Here we discuss certain necessary constraints associated with the scattering problem and their impact upon the associated evolution equations. Moreover, the direct linearization of the nonlinear evolution equations and an algorithm to construct an N-soliton solution are given.

I. INTRODUCTION

The inverse spectral (or scattering) transform (IST) method is a well-established technique to solve and investigate certain nonlinear partial differential equations of evolution type, a number of which are physically relevant.[1]

Attention has been recently given to the intermediate long wave (ILW) equation[2–7] because it brings into the field some novelty: that is, it is an integrodifferential, rather than purely differential, nonlinear equation, that is, integrable via a spectral problem based on a differential Riemann–Hilbert (RH) boundary value problem rather than an ordinary differential equation. Moreover, the ILW equation depends on a parameter which we call η, in such a way as to coincide, as η vanishes, with the Korteweg–de Vries (KdV) equation,[8] and, as η goes to infinity, with the Benjamin–Ono equation.[9–11]

In analogy with the well-known connection between the Korteweg–de Vries equation and the modified Korteweg–de Vries equation, a modified ILW equation (whose $\eta \to 0$ limit is the modified KdV equation) has also been introduced and investigated.[12,13]

Further progress in this direction has been made by extending[14] the class of intermediate-type long-wave equations, and by introducing[15,16] an intermediate version of the Kadomtsev–Petviashvili equation[17] (whose $\eta \to 0$ limit is of course the Kadomtsev–Petviashvili equation).

More recently,[18] a class of matrix nonlinear integral evolution equations was generated through the following 2×2 matrix spectral problem:

$$\psi^-(x,z) = G(x,z)\psi^-(x,z), \quad x\in\mathbb{R}, \tag{1a}$$

$$G(x,z) \equiv I + z\sigma_3 + U(x), \tag{1b}$$

where I is the identity matrix, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, z plays the role of spectral parameter, and $U(x)$ is a (complex) z-independent potential function.

Given the matrix function $U(x)$, (1) defines a homogeneous RH boundary value problem on a strip of the complex x plane. The matrices $\psi^\pm(x,z)$ are the boundary values of a function $\Psi(x,z)$ holomorphic in the horizontal strip between $\operatorname{Im} x = 0$ and $\operatorname{Im} x = \eta$:

$$\psi^-(x,z) \equiv \lim_{y\downarrow 0} \Psi(x+iy,z), \quad x\in\mathbb{R}, \tag{2a}$$

$$\psi^-(x,z) \equiv \lim_{y\uparrow\eta} \Psi(x+iy,z), \quad x\in\mathbb{R}. \tag{2b}$$

It turns out that $\Psi(x,z)$ can be written as

$$\Psi(x,z) = \exp\Big[-i\zeta(z)x\Big]\Big(\frac{1}{2i\eta}\int_{-\infty}^{\infty} \coth\Big[\frac{\pi}{\eta}(x'-x)\Big] \times h(x',z)dx' + \text{const}\Big), \quad 0 < \operatorname{Im} x < \eta, \tag{3}$$

where

$\exp[\eta\zeta(z)] = I + z\sigma_3$ and $h(x,z)$, defined by

$$h(x,z) = -(I+z\sigma_3)^{-1}\exp[i\zeta(z)x]\,U(x)\psi^+(x,z) \tag{4}$$

is Hölder continuous on $x\in\mathbb{R}$ and satisfies the condition $|\int_{-\infty}^{\infty} h(x,z)dx| < \infty$. Moreover, formula (3) implies the following periodicity condition:

$$\psi^-(x,z) = (E\psi^+)(x,z), \tag{5a}$$

where $E = \exp(i\eta\,\partial_x)$ is the formal shift operator

$$(Ef)(x) = f(x+i\eta). \tag{5b}$$

It was shown in Ref. 18 that the linear problem (1) and the associated class of evolution equations reduce, in the limit $\eta \to 0$, to the generalized Zakharov–Shabat–AKNS scattering problem[19] and to the associated class of nonlinear evolution equations.[19,20] Moreover, for the class of nonlinear equations associated with (1), an infinite family of conservation laws was derived and only elementary properties of the spectral problem were essential for that derivation. In fact, the emphasis in Ref. 18 was mainly on the novel nonlinear evolution equations, such as an intermediate version of the nonlinear Schrödinger equation, and on their associated Lax pair.

In this paper we present new results concerning the RH boundary value problem (1) and the class of evolution equations associated with it.

II. THE BASIS CONSTRAINTS

In the theory of matrix RH problems[21] of the type (1) an important role is played by the determinant of $G(x,z)$. In our case

$$\det G(x,z) = 1 - z^2 - z\,\mathrm{tr}(\sigma_3 U(x)) + \mathrm{tr}\,U(x) + \det U(x). \tag{6}$$

All the results of this paper are derived when the potential matrix $U(x)$ is subjected to the following two scalar constraints:

$$\mathrm{tr}(\sigma_3 U(x)) = 0, \tag{7a}$$

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$$\operatorname{tr} U(x) + \det U(x) = 0, \tag{7b}$$

or equivalently

$$U(x) = \sqrt{1 - Q^2(x)} - 1 + Q(x), \tag{8}$$

where $Q(x)$ is the off-diagonal part of $U(x)$. In this case the determinant of $G(x,z)$ takes the particularly simple form

$$\det G(x,z) = 1 - z^2, \tag{9}$$

independent of x with the following important consequences.

(i) The matrix $G(x,z)$ is invertible for every $x \in \mathbb{R}$; this is a necessary condition for the solvability of (1).

(ii) The total index κ of the matrix RH problem (1) is zero, since

$$\kappa = (2\pi)^{-1}[\arg(\det G(x,z))]_{-\infty}^{\infty}, \tag{10}$$

where $[\theta(x)]_{-\infty}^{\infty} = \theta(\infty) - \theta(-\infty)$.

Then an important theorem due to Gohberg and Krein[22] shows that "generically" the two partial indices κ_1, κ_2 ($\kappa = \kappa_1 + \kappa_2$) are both zero. This fact guarantees the existence and uniqueness of a bounded fundamental matrix $\Psi(x,z)$ associated with (1).

III. THE REDUCED CLASS OF EVOLUTION EQUATIONS

The existence and uniqueness of bounded solutions of (1) can be used in the construction of the IST method for the class of evolution equations introduced in Ref. 18, if and only if the constraints (7) are compatible with the evolution equations themselves. It will be shown in the following that this is indeed the case. Hence the constraints (7), introduced as requirements *for the solvability of (1), are in fact a reduction of* the class of equations introduced in Ref. 18 to the following class of matrix nonlinear evolution equations:

$$Q_t = \sigma_3 \gamma(\mathscr{L}) Q, \tag{11}$$

where

$$\mathscr{L} F \equiv i\sigma_3 (\sqrt{1 - Q^2}\, \mathscr{D} F + \tfrac{1}{2} Q \mathscr{D}^{-1}([Q,F]/\sqrt{1 + Q^2})), \tag{12}$$

F is off-diagonal, $\gamma(y)$ is an arbitrary polynomial in y, and

$$(\mathscr{D} f)(x) \equiv \frac{1}{\eta} \int_{-\infty}^{\infty} dy \left\{\sinh\left[\frac{\pi}{\eta}(y - x)\right]\right\}^{-1} f(y), \tag{13a}$$

$$(\mathscr{D}^{-1} f)(x) \equiv -\frac{1}{\eta} \int_{-\infty}^{\infty} dy \coth\left[\frac{\pi}{\eta}(y - x)\right] f(y). \tag{13b}$$

In order to show that (8) is a reduction for the class of evolution equations associated with (1), one has to show that the set of the matrices U satisfying (8) is closed with respect to the elementary deformations $\delta U^{(n)}$ such that $\delta\psi^{\pm} = B^{\pm}\psi^{\pm}$. Namely, one has to prove that if $U(x)$ satisfies (8) then

$$\delta U^{(n)} = (2\sqrt{1 - Q^2})^{-1}\{Q,\delta Q^{(n)}\} + \delta Q^{(n)}. \tag{14}$$

In Ref. 18 it was shown that the elementary deformations $\delta U^{(n)}$, such that $\delta\psi^{\pm} = B^{\pm}\psi^{\pm}$, are given by

$$\delta U^{(n)} = b_n\{((E - 1)L^n\sigma_3)(1 + U) + [L^n\sigma_3, U]\}, \tag{15}$$

with

$$B^{\pm}(x,z) = b_n \sum_{j=0}^{n} L^{n-j}\sigma_3 z^j, \quad b_n \text{ arbitrary constants}, \tag{16}$$

where $L^n\sigma_3$, defined in Ref. 18, is written here in the following more convenient form:

$$L^n\sigma_3 \equiv -\tfrac{1}{2}(i\mathscr{D}^{-1} + 1)F_n - \tfrac{1}{2}(i\mathscr{D} - 1)G_n, \tag{17}$$

in terms of the diagonal and off-diagonal matrices $F_n(x)$ and $G_n(x)$, respectively, which are constructed through the following recursion relations:

$$F_{n+1} = -\sigma_3((1 + U_c)F_n + \tfrac{1}{2}\{\mathscr{D} G_n, Q\} - \tfrac{1}{2}[G_n, Q]), \tag{18a}$$

$$G_{n+1} = \sigma_3(\tfrac{1}{2}\{F_n, Q\} - (i/2)[\mathscr{D}^{-1}F_n, Q] + (i/2)\{\mathscr{D} G_n, (1 + U_c)\} - \tfrac{1}{2}[G_n, U_c]), \tag{18b}$$

$$F_1 = 0, \quad G_1 = 2Q, \tag{18c}$$

where $[\ ,\ ]$ and $\{\ ,\ \}$ are the usual commutator and anticommutator between matrices and U_c is the diagonal part of U. The class of evolution equations is obtained by replacing $\delta U^{(n)}/b_n$ by U_t.

Using Eqs. (15), (17), and (8), one can show that

$$\delta U^{(n)} - \{(2\sqrt{1 + Q^2})^{-1}Q, \delta Q^{(n)}\} - \delta Q^{(n)} = -\frac{Q^2(\operatorname{tr} F_n)}{2\sqrt{1 + Q^2}} + \sqrt{1 + Q^2} \times\left(F_n - \frac{1}{2\sqrt{1 + Q^2}}[Q,G_n]\right). \tag{19}$$

Moreover, if (8) holds, one can prove by induction that the recursion equations (18) decouple in the following way:

$$F_n = (2\sqrt{1 + Q^2})^{-1}[Q,G_n], \tag{20a}$$

$$G_{n+1} = \mathscr{L} G_n, \quad G_1 = 2Q. \tag{20b}$$

Then, from (19) and (20a), one immediately gets

$$\delta U^{(n)} = (2\sqrt{1 + Q^2})^{-1}\{Q,\delta Q^{(n)}\} + \delta Q^{(n)}. \tag{21}$$

From (15), (17), and (20) one finally obtains the evolution equations (11) and from (16), (17), and (20) one gets the corresponding time evolution of function ψ,

$$\psi_t^{\pm} = \frac{\alpha_n}{2}\left(\sum_{j=0}^{n-1} z^j L^{n-j}\sigma_3 + z^n\sigma_3\right)\psi^{\pm}, \tag{22}$$

where

$$L^j\sigma_3 = -\tfrac{1}{2}(i\mathscr{D}^{-1} + 1)((\sqrt{1 + Q^2})^{-1}[Q, \mathscr{L}^{j-1}Q]) - (i\mathscr{D} - 1)\mathscr{L}^{j-1}Q \tag{23}$$

and the polynomial $\gamma(y)$, introduced in (11), is taken to be $\gamma(y) \equiv \alpha_n y^n$.

The first three equations of the class (11) are (see Ref. 18) the following:

(i) an intermediate wave equation

$$\gamma(y) = -icy, \quad U = \begin{pmatrix} u - 1 & \rho v \\ v & u - 1 \end{pmatrix}, \quad \rho \in \mathbb{R}, \tag{24a}$$

$$v_t = c\sqrt{1 + \rho v^2}\,\mathscr{D} v, \quad u = \sqrt{1 + \rho v^2};$$

(ii) an intermediate nonlinear Schrödinger equation

$$\gamma(y) = icy^2, \quad U = \begin{pmatrix} u - 1 & i\rho\psi^* \\ i\psi & u - 1 \end{pmatrix}, \quad \rho \in \mathbb{R}, \tag{24b}$$

$$i\psi_t - c[\sqrt{1 - \rho|\psi|^2}\,\mathscr{D}(\sqrt{1 - \rho|\psi|^2}\,\mathscr{D}\psi)$$

$$-\rho\psi\mathscr{D}^{-1}\mathrm{Re}(\psi^*\mathscr{D}\psi)] = 0,$$

$$u = \sqrt{1-\rho|\psi|^2};$$

(iii) an intermediate modified KdV equation

$$\gamma(y) = icy^3, \quad U = \begin{pmatrix} u-1 & \rho v \\ v & u-1 \end{pmatrix}, \quad \rho\in\mathbf{R},$$

$$v_t = c\sqrt{1-\rho v^2}\mathscr{D}(\sqrt{1-\rho v^2}\mathscr{D}(\sqrt{1-\rho v^2}\mathscr{D}v)\mathscr{D}^{-1}v\mathscr{D}v),$$

$$u = \sqrt{1-\rho v^2}. \tag{24c}$$

Taking the $\eta\to 0$ limit of Eqs. (24a)–(24c) one obtains the linear wave equation, the nonlinear Schrödinger equation, and the modified KdV equation, respectively.

The limit $\eta\to\infty$ can be immediately performed,[18] replacing $\mathscr{D}$ and $\mathscr{D}^{-1}$ by H and $-H$, respectively, where

$$(Hf)(x) \equiv \frac{1}{i\pi}\int_{-\infty}^{\infty} dy(y-x)^{-1}f(y) \tag{25}$$

is the Hilbert transform. We conclude this section by noticing that Eq. (24a) can be written in the following simple and suggestive form:

$$\mathscr{D}^{-1}\theta_t = c\sin\theta, \quad \theta = \theta(x,t), \tag{26}$$

where $v(x,t) = i\sin\theta(x,t)$. In the limit $\eta\to\infty$ Eqs. (26) become

$$H\theta_t = -c\sin\theta, \tag{27}$$

which we refer to as the sine-Hilbert equation, in analogy with the sine-Gordon equation θ_{xt} sin θ.

IV. THE DIRECT LINEARIZATION

Postponing to a separate paper the presentation of the IST method for the solution of the Cauchy problem associated with Eq. (11), we now present the direct linearization (DL)[23,24] for the class (11).

The DL is an algebraic approach based on the existence of a linear integral equation which provides a large class of solutions of the evolution equations (11).

Proposition: Let $\mu^{\pm}(x,t,z)$ be the solutions of the integral equations

$$\mu^{\pm}(x,t,z) - \int_l \mu^{\pm}(x,t,z')R^{\pm}(x,t,z')\frac{d\lambda(z')}{z'-z} = I, \tag{28}$$

where l and $d\lambda(z)$ are an arbitrary contour and measure

$$R^{\pm}(x,t,z) \equiv \psi_0^{\pm}(x,t,z)A(z)(\psi_0^{\pm}(x,t,z))^{-1},$$

$$A(z) \text{ arbitrary}, \tag{29}$$

and $(\psi_0^{\pm}(x,t,z),\ U(x,t))$ is a given solution of (1) and (22) (where, of course, ψ and U are replaced by ψ_0 and U_0). Assuming that the homogeneous version of (28) has only the trivial solution, then the matrices $\psi^{\pm}$, defined through

$$\psi^{\pm}(x,t,z) \equiv \mu^{\pm}(x,t,z)\psi_0^{\pm}(x,t,z), \tag{30}$$

solve Eq. (1) if the potential $U(x,t)$ is given by

$$U(x,t) = U_0(x,t) - \int_l [\mu^-(x,t,z)R^-(x,t,z)\sigma_3 - \sigma_3\mu^+(x,t,z)R^+(x,t,z)]d\lambda(z). \tag{31}$$

The proof is direct and as in the spirit of the method as it was introduced in Ref. 23; the constructive procedure used to obtain Eqs. (29) and (31) is illustrated in detail in Ref. 24.

V. N-SOLITON SOLUTION

The N-soliton solution for the class (11) can be obtained by setting $U_0(x,t) = 0$ (and then $\psi_0^{\pm}(x,t,z) = \exp[-i\zeta(z)x + (\alpha_n/2)z^n\sigma_3 t])$,

$$A_{lm}(1-\delta_{lm})\theta((-1)^l\mathrm{Im}\, z) \tag{32a}$$

[$\theta(x)$ is the usual step function], and

$$d\lambda(z) \equiv \begin{cases} \sum_{j=1}^{N} c_j\,\delta(z-z_j), & \mathrm{Im}\, z_j > 0, \quad \mathrm{Im}\, z > 0, \\ -\sum_{j=1}^{N} \bar{c}_j\,\delta(z-\bar{z}_j), & \mathrm{Im}\, \bar{z}_j < 0, \quad \mathrm{Im}\, z < 0. \end{cases} \tag{32b}$$

In this case Eq. (28) reduces to a $2N$th-order algebraic system; in particular, if $N = 1$ we have the following one-soliton solution:

$$u_{11}(x,t) = u_{22}(x,t) = (\bar{z}_1 - z_1)\sinh[\eta(k_1-\bar{k}_1)]/d(x,t), \tag{33a}$$

$$u_{12}(x,t) = \bar{c}_1 e^{\eta\bar{k}_1 - \gamma_1}[\cosh(\eta\bar{k}_1)e^{-\phi_-(k_1)} + \cosh(\eta k_1)e^{-\phi_-(\bar{k}_1)}]/d(x,t), \tag{33b}$$

$$u_{21}(x,t) = c_1 e^{-\eta k_1 - \gamma_1}[\cosh(\eta\bar{k}_1)e^{\phi_-(k_1)} + \cosh(\eta k_1)e^{\phi_-(\bar{k}_1)}]/d(x,t), \tag{33c}$$

where

$$\bar{z}_1 \equiv z(\bar{k}_1) = \tan(\eta\bar{k}_1), \quad z_1 \equiv z(k_1) = \tan(\eta k_1), \tag{34a}$$

$$d(x,t) \equiv \cosh[\eta(k_1-\bar{k}_1)] + \cosh[\phi_+(k_1) - \phi_-(\bar{k}_1)], \tag{34b}$$

$$\phi_{\pm}(k) \equiv 2ikx - \alpha_n(z(k))^n t \pm \gamma_1, \tag{34c}$$

$$-c_1\bar{c}_1/(z_1-\bar{z}_1)^2 \equiv e^{2\gamma_1 + \eta(k_1 - \bar{k}_1)}, \tag{34d}$$

and $k_I \equiv \frac{1}{2}\mathrm{Im}(k_1-\bar{k}_1)$ and $k_R \equiv \frac{1}{2}\mathrm{Re}(k_1-\bar{k}_1)$ satisfy the inequality

$$k_I^2 - k_R^2 - (\pi/2\eta)k_I < 0. \tag{35}$$

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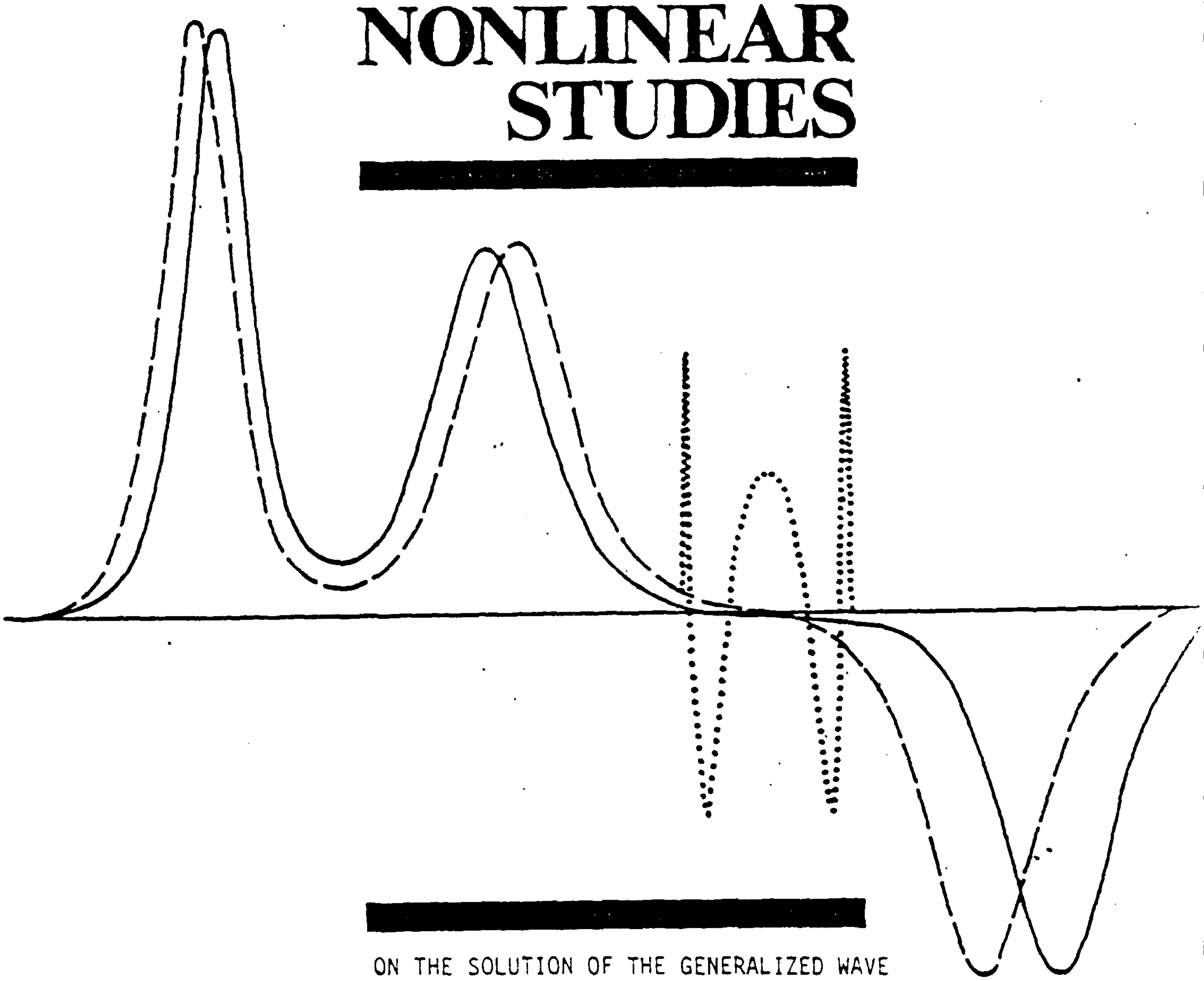

ON THE SOLUTION OF THE GENERALIZED WAVE AND GENERALIZED SINE-GORDON EQUATIONS

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Abstract

The generalized wave equation and generalized Sine-Gordon equations are known to be natural multidimensional differential geometric generalizations of the classical two dimensional versions. In this paper we associate a system of linear differential equations with these equations and show how the direct and inverse problems can be solved for appropriately decaying data on suitable lines. An "initial-boundary value" problem is solved for these equations.

In 1967 Gardner, Greene, Kruskal and Miura [1] discovered that the Cauchy problem, with suitably decaying initial data on the line, associated with the Korteweg-deVries (KdV) equation could be solved by making use of ideas from the theory of scattering/inverse scattering. Subsequently a number of nonlinear equations of physical interest have been solved by variants of this method, often referred to as the Inverse Scattering Transform (I.S.T.). Accounts of these techniques, associated algebraic structure and amenable nonlinear equations can be found in texts on this subject (see for example [2]).

An equation which fits into this framework is the Sine-Gordon equation:

$$u_{tt} - u_{xx} - \kappa \sin u = 0. \tag{1.1}$$

The Sine-Gordon equation is of interest to physicists and mathematicians. It was first solved by I.S.T. in [3], In physics it arises in the study of Josephson junctions, particle physics, stability of fluid motions etc. In mathematics it has arisen classically in the study of differential geometry. In this paper we will describe a method which enables us to carry out the I.S.T. for certain nonlinear n dimensional generalizations of the Sine-Gordon and wave equations ($\kappa = 0$) which arise in the study of differential goemetry.

Originally the Sine-Gordon equation was derived in the study of surfaces of constant negative curvature contained in Euclidean space $\mathbb{R}^3$. There is an intimate connection between such surfaces and solutions of the equation. Indeed in 1875 Bäcklund [4] considered the following. Let M and $\bar{M}$ be surfaces in $\mathbb{R}^3$ and $\ell: M \to \bar{M}$ be a diffeomorphism such that for any point p in M and corresponding point $\bar{p} = \ell(p)$ one has the following.

(a) The line determined by p, and $\bar{p}$ is tangent to M and $\bar{M}$ at p and $\bar{p}$ respectively;

(b) the distance $d(p,\bar{p}) = r>0$ is a constant independent of p;

(c) the angle between the normal vectors $N(p)$ and $\bar{N}(\bar{p})$ to the surfaces is a constant θ independent of p.

Bäcklund proved that under these conditions the surfaces M and $\bar{M}$ have constant Gaussian curvature $\kappa = \bar{\kappa} = -\sin^2\theta/r^2$ which can be normalized to be -1. Moreover he showed that given any surface $M \subset \mathbb{R}^3$ with curvature $\kappa = -1$ there exists a two parameter family of surfaces $\bar{M}$ with curvature $\bar{\kappa} = -1$ related to M by diffeomorphisms which satisfy (a)-(c).

The analytic interpretation of these results originated in what is now called a Bäcklund transformation, which provides new solutions to the Sine-Gordon equation from a given one. Later Bianchi [5] obtained a permutability theorem for surfaces which provides superposition formulae for the Sine-Gordon equation.

Motivated in part by the work of [6] the natural geometric generalizations of these results were obtained in [7,8] by considering hyperbolic (constant sectional curvature equal to -1) n-dimensional submanifolds M^n of the Euclidean space $\mathbb{R}^{2n-1}$. The geometric results for hyperbolic manifolds M^n contained in $\mathbb{R}^{2n-1}$ were extended [9] to manifolds M^n of constant sectional curvature $\kappa<1$ (resp. $\kappa<-1$) contained in the unit spheres S^{2n-1} (resp. hyperbolic space H^{2n-1}). In particular, the zero-curvature submanifolds of the unit sphere correspond to solutions of a generalized wave equation (GWE) which is a homogeneous version of the generalized Sine-Gordon equation (GSGE) associated with embeddings in Euclidean space.

The higher dimensional version of Bäcklund's results takes the following form:

$$dX + XA^tX = A - XB \tag{1.2}$$

where

$$dX = \sum_{j=1}^{n} \frac{\partial X}{\partial x_j} dx_j$$

$$A_{ij} = \beta_i(z)\, a_{ij}\, dx_j$$

$$B_{ij} = \frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i} dx_j - \frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j} dx_i \quad 1 \le i,\, j \le n \tag{1.3}$$

and $a = \{a_{ij}\} \in R^{n\times n}$. (1.2-1.3) reduce to the Bäcklund transformation for the generalized Sine-Gordon equation (GSGE) when

$$\beta_i(z) = (z^2 + (2\delta_{i1} - 1))/_{2z} \tag{1.4}$$

and for the generalized wave equation (GWE) when

$$\beta_i(z) = -(1-z^2)/_{2z} \equiv \lambda(z). \tag{1.5}$$

The compatibility condition required for the existence of solutions to these Bäcklund transformations results in a system of second order partial differential equations for an orthogonal nxn matrix $a = \{a_{ij}\}$ in (1.2) which is a function of n independent variables $a = a(x_1, x_2, \cdots x_n)$. The equation has the form.

$$\frac{\partial}{\partial x_i}\left(\frac{1}{a_{1i}} \frac{\partial a_{1j}}{\partial x_i}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j}\right)$$
$$+ \sum_{k\neq i,j} \frac{1}{a_{1k}^{\;2}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1j}}{\partial x_k} = \varepsilon a_{1i}\, a_{1j}, \quad i\neq j$$
$$\frac{\partial}{\partial x_k}\left(\frac{1}{a_{1j}} \frac{\partial a_{1i}}{\partial x_j}\right) = \frac{1}{a_{1k} a_{1j}} \frac{\partial a_{1i}}{\partial x_k} \frac{\partial a_{1k}}{\partial x_j}, \quad i,j,\ k \text{ distinct}$$

$$\frac{\partial a_{jk}}{\partial x_i} = \frac{a_{ji}}{a_{1i}} \frac{\partial a_{1k}}{\partial x_i}, \quad i\neq k, \tag{1.6}$$

where $\varepsilon=1$ for the GSGE and $\varepsilon=0$ for the GWE.

We observe that when $n=2$ and $\varepsilon=1$ (GSGE), the orthogonal matrix $a = \{a_{ij}\}$:

$$a = \begin{pmatrix} \cos\frac{u}{2} & \sin\frac{u}{2} \\ -\sin\frac{u}{2} & \cos\frac{u}{2} \end{pmatrix} \tag{1.7}$$

for the function $u=u(x,t)$ reduces the GSGE to the classical Sine-Gordon equation (1.1). We note also that if the parameter z in (1.2) is given by $z=\tan\theta/2$ then θ is the constant in Bäcklund's statement (c) above. On the other hand when $n=2$ and $\varepsilon=0$, then with (1.7) the GWE reduces to the wave equation (1.1) with $\kappa=0$. When $n\geqslant 3$ the generalization of the wave equation discussed here is nonlinear. A Bäcklund transformation and a superposition formula for the GWE was obtained in [9].

The Bäcklund transformations (1.2) described above, are in fact matrix Riccati equations. Linearizations of such a system can be performed in a straightforward manner (see for example [10]). Introducing the transformation,

$$X = UV^{-1} \tag{1.8}$$

where U,V are $n\times n$ matrix functions of $x_1,\dots x_n$ the following linear system is deduced,

$$\begin{pmatrix} dU \\ dV \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{1.9}$$

with the components of A, B given in (1.3). Compatibility ensures that the orthogonal matrix $a = \{a_{ij}\}$ satisfies the GSGE with (1.4) and GWE with (1.5). Alternatively if we call $\begin{pmatrix} U \\ V \end{pmatrix} = \psi$, the following linear system of 2n ode's are obtained:

$$\frac{\partial\psi}{\partial x_j} = \lambda \tilde{A}_j \psi + C_j \psi \tag{1.10}$$

where $\tilde{A}_j$, C_j are $(2n\times 2n)$ matrices with the block structure

$$\tilde{A}_j = \begin{pmatrix} 0 & \tilde{a}_j \\ \tilde{a}_j^t & 0 \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}. \qquad (1.11)$$

Here $\tilde{a}_j$, $\tilde{\gamma}_j$ are $(n \times n)$ matrices having the following structure

$$\tilde{a}_j = (\tfrac{\delta}{\lambda} - 1)e_1 \, a_j + a_j, \qquad (1.12)$$
$$a_j = ae_j$$

where $e_j = \{e_j\}_{ik}$ is the unit matrix

$$\{e_j\}_{ik} = \begin{cases} 1 & i = k = j \\ 0 & \text{otherwise} \end{cases}, \qquad (1.13)$$

and in component form γ_j takes the form:

$$(\gamma_j)_{k\ell} = (1-\delta_{kj}) \frac{1}{a_{1k}} \frac{\partial a_{1j}}{\partial x_k} \delta_{\ell j} - (1-\delta_{\ell j}) \frac{1}{a_{1\ell}} \frac{\partial a_{1j}}{\partial x_\ell} \delta_{kj} \qquad (1.14).$$

In (1.12) a is the orthogonal matrix: $\mathbb{R}^n \to SO(n)$ associated with the GWE when $\delta = \lambda$ and with GSGE when $\delta = \frac{1}{2}(z+\frac{1}{z})$, $\lambda = \frac{1}{2}(z-\frac{1}{z})$, and γ_j is the matrix (1.14): $\mathbb{R}_n \to M_n(\mathbb{R})$ $\gamma_j + \gamma_j^t = 0$. Although γ_j is determined by a, it will be convenient to treat $(\bar{a}, \gamma_1, \ldots \gamma_n)$ as the data. Then both (1.6) and (1.14) arise as the compatibility conditions for the scattering problem (1.10).

Since we shall separately examine the two cases GSW and GSGE, we write down the explicit scattering problems which are compatible with each of these equations.

For the GWE the scattering problem takes the form; $\psi = \psi(x,\lambda)$:

$$\frac{\partial\psi}{\partial x_j} = \lambda A_j\psi + C_j\psi \tag{1.15}$$

with

$$A_j \equiv \begin{pmatrix} 0 & a_j \\ a_j^t & 0 \end{pmatrix}, \tag{1.16}$$

and e_j is given in (1.14) and C_j given by (1.11, 1.14).

For the GSGE the scattering problem for $\psi = \psi(x,z)$ is

$$\frac{\partial\psi}{\partial x_j} = \delta(z) \begin{pmatrix} 0 & e_1 a_j \\ a_j^t e_1 & 0 \end{pmatrix}\psi$$

$$+ \lambda(z) \begin{pmatrix} 0 & (I-e_1)a_j \\ a_j^t(I-e_1) & 0 \end{pmatrix}\psi + C_j\psi, \tag{1.17a}$$

$\delta(z)$, $\lambda(z)$, C_j given above, or equivalently

$$\frac{\partial\psi}{\partial x_j} = \frac{z}{2} A_j\psi + \frac{z}{2} B_j\psi + C_j\psi, \tag{1.17b}$$

where

$$B_j = \begin{pmatrix} 0 & ua_j \\ a_j^t u & 0 \end{pmatrix}, \quad u = \mathrm{diag}(+1,-1,\ldots.-1)$$

In this paper we show how the direct and inverse scattering problems associated with the GWE: (1.15) and the GSGE: (1.17) can be solved for matrix potentials tending to the identity sufficiently fast in certain "generic" directions (to be discussed later). It is along such directions (lines) that suitable initial values for the entries of $a(x)$ and the matrices $\gamma_j(x)$ can be specified. In §2-4 the analysis for the GWE is given and in §5-8 the analogous problems are discussed for the GSGE.

Finally, we remark that solving the n dimensional GWE and GSGE reduces to the study of the scattering/inverse scattering associated with a coupled system of n one dimensional ode's. This is in marked contrast to other attempts to isolate solvable (local) multidimensional nonlinear evolution equations which are the compatibility condition of two Lax type operators

$$L\psi = \lambda\psi \tag{1.18}$$

$$\psi_t = M\psi \tag{1.19}$$

where L is a partial differential operator with the variable t entering only parametrically. Although nonlinear evolution equations in three independent variables can be associated with suitable Lax pairs (e.g. the Kadomtsev-Petviashvili, Davey-Stewartson and three-wave interaction equations - see for example the review [11]), little progress has been made in more than three independent variables. In this context one has to overcome a serious constraint inherent in the scattering theory for higher dimensional partial differential operators in order to be able to find associated solvable nonlinear equations: i.e. the scattering data generally satisfies a nonlinear equation [see 12-14]. The analysis discussed herein completely avoids such problems since the linear system is simply a compatible set of n linear one dimensional scattering problems. On the other hand, these results demonstrate that the initial value problem is posed with given data along lines and not on (n-1) dimensional manifolds.

§2 The Forward Problem for the GWE.

We consider here the spectral problem (1.15), assuming the associated compatibility conditions, i.e. the GWE. The strategy is to transform (1.15) to a standard form and to associate to it a Riemann-Hilbert factorization problem as in [15]. The transformation uses the 2nx2n orthogonal matrices

$$U_1 = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \; U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad U = U_1 U_2. \tag{2.1}$$

If ψ is a fundamental matrix solution of (1.15) then the function

$$\tilde{\psi}(x,\lambda) = U(x)^{-1}\psi(x,\lambda) \tag{2.2}$$

satisfies

$$\frac{\partial\tilde{\psi}}{\partial x_j} = \lambda J_j \tilde{\psi} + Q_j \tilde{\psi} \tag{2.3}$$

where

$$J_j = U^{-1}A_j U = U_2^{-1} \begin{pmatrix} 0 & e_j \\ e_j & 0 \end{pmatrix} U_2 = \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix}, \tag{2.4}$$

and

$$Q_j = U^{-1}C_j U - U^{-1} \frac{\partial}{\partial x_j} U = U_2^{-1} \begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2 \tag{2.5}$$

where

$$\alpha_j = -a^t \frac{\partial a}{\partial x_j} \tag{2.6}$$

Conversely, (2.2)-(2.6) imply that ψ is a solution of (1.15). We look for a solution $\tilde{\psi}$ in the form

$$\tilde{\psi}(x,\lambda) = m(x,\lambda)e^{\lambda x \cdot J}, \; x \cdot J = \sum_{j=1}^{n} x_j J_j. \tag{2.7}$$

Then (2.3) is equivalent to

$$\frac{\partial m}{\partial x_j} = \lambda[J_j, m] + Q_j m. \tag{2.8}$$

These equations imply that det m is constant. We look for m such that

$$m(\cdot,\lambda) \text{ is bounded; } \det m(x,\lambda) \equiv 1. \qquad (2.9)$$

Proposition (2.1). Suppose that for some $\lambda \in \mathbb{C}$, m_1 and m_2 are two solutions of (2.8), (2.9). Then there is a matrix $W(\lambda) \in SL(2n,\mathbb{C})$ such that

$$m_2(x,\lambda) \equiv m_1(x,\lambda) e^{\lambda x \cdot J} W(\lambda) e^{-\lambda x \cdot J}. \qquad (2.10)$$

Moreover, if $\lambda \notin i\,\mathbb{R}$ then W is diagonal.

Proof. One checks that

$$\frac{\partial}{\partial x_j} [e^{-\lambda x \cdot J} m_1(x,\lambda)^{-1} m_2(x,\lambda) e^{\lambda x \cdot J}] \equiv 0 \qquad (2.11)$$

so the matrix in brackets, $W(\lambda)$, is independent of x. Now (2.9) implies $\exp(\lambda x \cdot J) W(\lambda) \exp(-\lambda x \cdot J)$ is bounded with respect to x, which is only possible if $\lambda \in i\,\mathbb{R}$ or $W(\lambda)$ is diagonal.

We study the problem (2.8), (2.9) by restricting to lines in $\mathbb{R}^n$. Let w be a unit vector in $\mathbb{R}$ and y a vector orthogonal to w. Along the line

$$L(w,y) = \{ y + sw : s \in \mathbb{R} \} \qquad (2.12)$$

we consider the restriction of m:

$$\tilde{m}(s,\lambda) \equiv m(s,\lambda;w;y) \equiv m(y+sw,\lambda).$$

Then (2.8) gives

$$\frac{\partial \tilde{m}}{\partial s} = \lambda [J_w, \tilde{m}] + Q\tilde{m};$$

$$J_w \equiv w \cdot J \equiv \Sigma w_j J_j,$$

$$Q(s) \equiv Q(s,w,y) \equiv \Sigma w_j Q_j (y+sw). \qquad (2.14)$$

Definition (2.1).

The data $\{\alpha_j, \gamma_j\}$ is small in the direction w if the operator norm of the

associated matrix function Q satisfies

$$\int_{-\infty}^{\infty} \| Q(s,w,y) \| \, ds \leqslant k < 1 \tag{2.15}$$

for some constant k and all y orthogonal to w.

Definition (2.2).

The data $\{\alpha_j, \gamma_j\}$ is *asymptotically flat* in the direction w if each derivative of each entry of the matrices α_j, γ_j is rapidly decreasing at infinity on each line $L(w,y)$, uniformly with respect to y. Thus, for each such matrix entry f, each integer $N > 0$, and each multi-index β,

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta} f(y+sw)\right| \leqslant C(1+|s|)^{-N} \tag{2.16}$$

for every $y \perp w$ and $s \in \mathbb{R}$.

Definition (2.3).

The direction w is *oblique* if the $2n$ numbers $\{\pm w_j\}$ are distinct.

Theorem (2.2).

Suppose the data $\{\alpha_j, \gamma_j\}$ is small and asymptotically flat in some oblique direction w. Then for each $\lambda \in \mathbb{C} \setminus i\,\mathbb{R}$ there is a unique $m(\cdot,\lambda)$ which solves the problem (2.8) and (2.9) and satisfies the asymptotic condition

$$\lim_{s \to -\infty} m(y+sw,\lambda) = I, \text{ all } y \perp w. \tag{2.17}$$

Moreover m is bounded, $m(s,\cdot)$ is holomorphic on $\mathbb{C} \setminus i\,\mathbb{R}$, and the limits

$$m_{\pm}(x,\lambda) = \lim_{\varepsilon \to 0+} m(x,\lambda \pm \varepsilon) \tag{2.18}$$

exist and are smooth functions on $\mathbb{R}^n \times i\,\mathbb{R}$. Also

$$\lim_{|\lambda| \to \infty} m(x,\lambda) = I, \tag{2.19}$$

uniformly with respect to x.

Before discussing the proof of this theorem, let us consider the implications. For $\lambda \in i\mathbb{R}$ the limits $m_\pm$ give two solutions of (2.8), (2.9). Therefore Proposition (2.1) implies the following.

Corollary (2.3).

There is a matrix-valued function $V : i\mathbb{R} \to SL(2n,\mathbb{C})$ such that

$$m_+(x,\lambda) = m_-(x,\lambda)e^{\lambda x\cdot J}V(\lambda)e^{-\lambda x\cdot J} \tag{2.20}$$

for all $x \in \mathbb{R}^n$, $\lambda \in i\mathbb{R}$.

Definition (2.4).

The function V is the *scattering data* associated to (a,γ_j) and the direction w.

We now sketch the proof of Theorem (2.2). Note that

$$\alpha_j + \alpha_j^t = -\frac{\partial}{\partial x_j}(a^t a) \equiv 0, \tag{2.21}$$

$$Q_j + Q_j^t = 0, \tag{2.22}$$

In particular, the diagonal entries of Q_j are zero. The problem (2.14) with the conditions

$$\tilde{m}(\cdot,\lambda) \text{ is bounded and } \lim_{s\to\infty} \tilde{m}(s,\lambda) = I \tag{2.23}$$

is exactly of the kind considered in [15]. Indeed $Q_{jj} \equiv 0$ and J_w is diagonal with distinct entries (since w is oblique). It follows from the results of [15] and the assumption (2.15) that (2.14), (2.23) has a unique solution $\tilde{m}$ which is bounded and holomorphic for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ and has a continuous limit on $\mathbb{R}^n \times i\mathbb{R}$. Moreover, $\tilde{m}$ is smooth with respect to s hence our assumptions

imply also that it is smooth with respect to y. These considerations give us many of the properties of m, which is defined by

$$m(y+sw,\lambda) = m(s,\lambda;w,y),\ y \perp w. \tag{2.24}$$

To show that m satisfies the full set of equations (2.8), we use the compatibility conditions (GWE). It is most convenient to choose new variables $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ by an orthogonal change of coordinates in $\mathbb{R}^n$ chosen such that $\frac{\partial}{\partial \tilde{x}_1} = \frac{\partial}{\partial s}$.

The desired equations (2.8) take the form

$$\frac{\partial m}{\partial \tilde{x}_j} = \lambda[J_j',m] + Q_j'm \equiv R_j'm \tag{2.25}$$

for $j > 1$, and

$$\frac{\partial m}{\partial \tilde{x}_1} = \frac{\partial m}{\partial s} = \lambda[J_w,m] + Qm \equiv Rm.$$

The compatibility conditions (GWE) imply

$$\frac{\partial Q}{\partial \tilde{x}_j} + QQ_j' = \frac{\partial Q_j'}{\partial s} + Q_j'Q,\ j > 1; \tag{2.26}$$

$$[J_j', Q] = [J_w,Q_j'],\ j > 1. \tag{2.27}$$

The solution to (2.14) satisfies the integral equations (see [15])

$$\tilde{m}(s,\lambda) = I + \int_{\pm\infty}^{s} \phi((s-t)\lambda)[Q(t)\tilde{m}(t,\lambda)]dt$$

where the limit $\pm\infty$ depends on the matrix entry and on the sign of $\mathrm{Re}\lambda$, while ϕ operates on matrices by

$$\phi(u)[B] = e^{uJ_w}Be^{-uJ_w} \tag{2.29}$$

We utilize (2.27) (employing shorthand notation) to compute

$$\frac{\partial m}{\partial x_j} - \lambda[J_j',m] = \int^s \phi\{\frac{\partial Q}{\partial \bar{x}_j}m + Q\frac{\partial m}{\partial \bar{x}_j} - \lambda[J_j',Qm]\}dt$$

$$= \int^s \phi\{\frac{\partial Q_j'}{\partial t}m + [Q_j',Q]m + Q\frac{\partial m}{\partial \bar{x}_j} - \lambda[J_w,Q_j']m - \lambda Q[J_j',m]\}dt$$

$$= \int^s \frac{d}{dt}\phi\{Q_j'm\}dt + \int^s \phi\{Q(\frac{\partial m}{\partial \bar{x}_j} - \lambda[J_j',m] - Q_j'm)\}dt$$

$$= Q_j'm + \int^s \phi\{Q(\frac{\partial m}{\partial x_j} - \lambda[J_j',m] - Q_j'm)dt. \tag{2.30}$$

Thus

$$\frac{\partial m}{\partial \bar{x}_j} - R_j'm = \int^s \phi[Q(\frac{\partial m}{\partial \bar{y}_j} - R_j'm)]dt, \tag{2.31}$$

which implies (2.25). (Note that the asymptotic conditions were used in the calculation (2.30), to eliminate a boundary term in the intergration.) This completes the proof of Theorem (2.2).

We turn now to the properties of the scattering data V. We introduce an automorphism of 2nx2n matrices:

$$B^\sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} B \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.32}$$

<u>Theorem (2.3)</u>.

The scattering data V has the following properties:

each entry of V-I belongs to the Schwartz space $S(i\mathbb{R})$; (2.33)

$$V(-\lambda) = V(\lambda)^t = \overline{V(\lambda)} = (V(\lambda)^\sigma)^{-1}. \tag{2.34}$$

Proof.

(2.33) follows from results in [15]. To obtain the symmetries (2.34), observe first that J_j and Q_j are real and

$$J_j^t = J_j = -J_j^\sigma, \quad Q_j = -Q_j^t = Q_j^\sigma. \tag{2.35}$$

It follows $\overline{m(x,\lambda)}$ satisfies the same equation as $m(s,\lambda)$ and that both $m(x,\lambda)^\sigma$ and $(m(x,\lambda)^{-1})^t$ satisfy the same equation as $m(x,-\lambda)$. The boundedness and asymptotic conditions are also satisfied, so

$$m(x,\bar{\lambda}) = \overline{m(x,\lambda)}, \tag{2.36}$$

$$m(x,-\lambda) = (m(x,\lambda)^{-1})^t = m(x,\lambda)^\sigma. \tag{2.37}$$

Therefore

$$\begin{aligned} V(-\lambda) &= m_-(0,-\lambda)^{-1} m_+(0,-\lambda) \\ &= m_+(0,\lambda)^t (m_-(0,\lambda)^{-1})^t = V(\lambda)^t, \end{aligned} \tag{2.38}$$

and similarly for the remaining symmetries.

Let us remark here that the construction of m by a Neumann series implies the estimates

$$\begin{aligned} &\|m\| \leq (1-k)^{-1}, \ \|m - I\| \leq k(1-k)^{-1} \\ &\|m^{-1}\| \leq (1-k)^{-1}, \ \|m^{-1} - I\| \leq k(1-k)^{-1}, \end{aligned} \tag{2.39}$$

where $k < 1$ is the constant of (2.15). It follows that

$$\|V - I\| \leq 2k(1-k)^{-2}.$$

In particular,

$$\|V - I\| \leq 1 \text{ if } 0 \leq k \leq 2 - \sqrt{3}.$$

We conclude this section with a brief discussion of normalizations and the relationship of this treatment of the forward problem to that in [15]. The normalization (2.17) depends on the choice of a direction w; therefore the solution m and the associated scattering data V depend on w. In [15], with $n=1$, the normalization was made at $-\infty$ and the resulting scattering data V had certain principal minors identically equal to 1. Here, the same considerations show that for a given direction w certain principal minors of the associated scattering data V are $\equiv 1$. In the absence of a single natural oblique direction, we have chosen to consider all possible scattering data and have not imposed conditions on principal minors. We return to this question at the end of Section 3.

§3 The Inverse Problem for the GWE.

Suppose $V : i\,\mathbb{R} \to SL(2n,\mathbb{C})$ is a matrix-valued function which satisfies the conditions (2.33) and (2.34). Suppose also that

$$\|V(\lambda) - I\| < 1, \quad \lambda \in i\,\mathbb{R}. \tag{3.1}$$

Theorem (3.1).

For each $x \in \mathbb{R}^n$ there is a unique matrix-valued function $m(x,\cdot)$ which is bounded and holomorphic on $\mathbb{C} \setminus i\,\mathbb{R}$, with continuous limits $m_\pm$ on $i\,\mathbb{R}$, and which satisfies

$$m_+(x,\lambda) = m_-(x,\lambda)e^{\lambda x\cdot J}V(\lambda)e^{-\lambda x\cdot J}, \lambda \in i\,\mathbb{R}$$

$$\lim_{|\lambda|\to\infty} m(x,\lambda) = I. \tag{3.2}$$

The function m is smooth on $\mathbb{R}^n x(\mathbb{C} \setminus i\,\mathbb{R})$ and satisfies a system of equations

$$\frac{\partial m}{\partial x_j} = \lambda[J_j,m] + Q_j(x)m \tag{3.3}$$

where $Q_j + Q_j^t \equiv 1$ and Q_j is real,

$$Q_j(x) = U_2^{-1}\begin{pmatrix} \alpha_j(x) & 0 \\ 0 & \gamma_j(x) \end{pmatrix} U_2. \tag{3.4}$$

Moreover, the data $\{\alpha_j,\gamma_j\}$ is asymptotically flat in every oblique direction in $\mathbb{R}^n$.

This theorem essentially follows from results in [15]. One way to obtain the equations (3.3) is to note that the function $n_j = \frac{\partial m}{\partial x_j} - \lambda[J_j,m]$ also satisfies the Riemann-Hilbert condition (3.2), from which it follows that

$Q_j = n_j m^{-1}$ is continuous across $i\mathbb{R}$. Therefore Q_j is entire; it is bounded, hence independent of λ, which gives (3.3). The symmetry conditions (2.34) imply that $m(x,\bar{\lambda})$, $(m(x,-\lambda)^{-1})^t$ and $m(x,-\lambda)^\sigma$ also solve the Riemann-Hilbert problem (3.2). By uniqueness, m has the symmetries (2.36) and (2.37). Therefore Q_j is real and has the symmetries (2.35), which in turn give (3.4). Finally, an oblique direction w corresponds to a diagonal matrix $J_w = \Sigma w_j J_j$ having distinct entries, and the results of [15] give rapid decrease of the data Q_j along lines in the direction w, as desired.

Remark. The data Q_j generally does not decrease rapidly in directions which are not oblique.

To connect this result to the GWE, we need one more step.

Lemma (3.2).

There is a function $a : \mathbb{R}^n \to SO(n)$ such that

$$\alpha_j = -a^t \frac{\partial a}{\partial x_j} \tag{3.5}$$

Proof.

The compatibility relations for the system (3.3) imply

$$\frac{\partial \alpha_j}{\partial x_k} + \alpha_j \alpha_k = \frac{\partial \alpha_k}{\partial x_j} + \alpha_k \alpha_j . \tag{3.6}$$

These in turn are the compatibility relations for (3.5). If a solves (3.5) then $\frac{\partial}{\partial x_j}(a^t a) \equiv 0$, so we can guarantee that $a \in SO(n)$ by choosing it to belong to SO(n) at a specified point or asymptotically in some oblique direction.

A solution of (3.5) is unique up to left multiplication by a fixed element of SO(n). If a is any such solution we refer to $\{a, \gamma_j\}$ as inverse data for the function V.

Theorem (3.3).

If $\{a,\gamma_j\}$ are inverse data for V, they satisfy the GWE.

Proof.

We simply reverse the procedure at the beginning of the preceding section. The function

$$\psi(x,\lambda) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} U_2 \ m(x,\lambda)e^{\lambda x \cdot J} \qquad (3.7)$$

satisfies the system (1.14), so (a,γ_j) satisfy the GWE.

Let us connect the inverse data explicitly to the asymptotics of m in λ. By [15], m has an asymptotic expansion

$$m(x,\lambda) \sim \sum_{\nu=0}^{\infty} m_\nu(x)\lambda^{-\nu}, \ \lambda\to\infty. \qquad (3.8)$$

This expansion can be differentiated term by term, giving

$$\frac{\partial}{\partial x_j} m_\nu = Q_j m_\nu + [J_j, m_{\nu+1}]. \qquad (3.9)$$

In particular, $m_0 \equiv I$ and so we obtain

$$\begin{aligned} Q_j(x) &= -[J_j, m_1(x)] \\ &= -\lim_{\lambda\to\infty} \lambda[J_j, m(x,\lambda)]. \end{aligned} \qquad (3.10)$$

This gives another method for deriving the symmetries (2.35) of Q from symmetries (2.36) and (2.37) of m.

As we noted at the end of section 2, different functions V may occur as scattering data for the same inverse data unless some further normalization is imposed. Therefore to complete the analysis of the relationship between solutions of the GWE and scattering data, we need to know when two functions V_1, V_2 as above give rise to the same inverse data. Let m_1, m_2 be the associated solutions of (3.2). If the inverse data is the same, then by Proposition (2.1),

$$m_2(x,\lambda) \equiv m_1(x,\lambda)\Delta(\lambda), \quad \lambda \in \mathbb{C} \setminus i\,\mathbb{R} \tag{3.11}$$

where Δ is diagonal and holomorphic and has boundary values $\Delta_\pm$; moreover $\Delta(\lambda) \to I$ as $|\lambda|\to\infty$. Δ has the same symmetry properties as m, so Δ is the solution of a Riemann-Hilbert problem (2.3) for a diagonal V. Clearly V_1 and V_2 are related by

$$V_2 = (\Delta_-)^{-1} V_1 \Delta_+. \tag{3.12}$$

In particular, V gives trivial inverse data if and only if V is diagonal. Conversely, if V_2 and V_1 are related by (3.12), where $\Delta_\pm$ are the boundary values of the solution to (2.3) for a diagonal V, then V_1 and V_2 have the same inverse data.

§4 A Well-posed Initial-Value Problem for the GWE.

The result of the preceding two sections both suggest and solve an "initial boundary" value problem for the GWE. Let us say that a solution $\{a,\gamma_j\}$ of the GWE is small if there is some oblique direction such that the associated data $\{\alpha_j,\gamma_j\}$ is both small and asymptotically flat in that direction. As before, if w is a direction (unit vector) in $\mathbb{R}^n$ and y is orthogonal to w, we parameterize the line $L(w,y)$ by $s \to y + sw$. Without loss of generality we may translate the coordinates and take $y = 0$.

Theorem (4.1).

Suppose w is an oblique direction in $\mathbb{R}^n$. Suppose $\tilde{a} : L(w,0) \to SO(n)$ and $\tilde{\gamma} : L(w,0) \to M_n(\mathbb{R})$ are smooth mappings such that $\tilde{\alpha} = -\tilde{a}^t \frac{\partial \tilde{a}}{\partial s}$ and $\tilde{\gamma}$ are Schwartz functions of s, $\tilde{\gamma}^t + \tilde{\gamma} \equiv 0$ and

$$\int_{-\infty}^{\infty} \| \tilde{\alpha}(s) \| \, ds < 3 - \sqrt{2}.$$

Then there is a unique small solution $\{a,\gamma_j\}$ of the GWE such that

$$\begin{aligned} \tilde{a}(s) &\equiv a(sw) \\ \tilde{\gamma}(s) &\equiv \Sigma w_j \gamma_j(sw) \end{aligned} \tag{4.1}$$

Proof.

Let $\tilde{m}$ be the solution of

$$\frac{\partial \tilde{m}}{\partial s}(s,\lambda) = \lambda[J_w,\tilde{m}] + Q\tilde{m}, \ \lim_{s\to-\infty} \tilde{m}(s,\lambda) = I \tag{4.2}$$

where $J_w = \Sigma w_j J_j$ and $Q = U_2^{-1}\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix} U_2$. There is a mapping $V: i\,\mathbb{R} \to SL(n,\mathbb{C})$ such that for $\lambda \in i\,\mathbb{R}$,

$$\tilde{m}_+(s,\lambda) = \tilde{m}_-(s,\lambda)e^{\lambda sJ_w}V(\lambda)e^{-\lambda sJ_w}. \tag{4.3}$$

Note the term $e^{\lambda sJ_w}V(\lambda)e^{-\lambda sJ_w}$ is the specialization to the line L(w,0) of $e^{\lambda x\cdot J}V(\lambda)e^{-\lambda x\cdot J}$. Thus factorization of this latter function gives us an extension to $\mathbb{R}^n$ of $\tilde{m}$. V satisfies the hypotheses of Theorem (3.1) so there is an associated solution m of the Riemann-Hilbert probelm (3.2) and

$$\tilde{m}(s,\lambda) \equiv m(sw,\lambda). \tag{4.4}$$

Let $\{a,\gamma_j\}$ be inverse data for V, normalized so that $a(s,w,y=0) = \tilde{a}(s)$ Because of (4.4) we obtain

$$\alpha(s) \equiv \Sigma w_j\alpha_j(sw)$$

$$\gamma(s) \equiv \Sigma w_j\gamma_j(sw). \tag{4.5}$$

The first identity implies $\frac{da}{as}a^t = \frac{d\tilde{a}}{ds}a^t$ on L(w,0) so we obtain $a \equiv \tilde{a}$ on L(w,0). This completes the proof of existence. Uniqueness follows from the fact that the scattering data associated to a small solution $\{a,\gamma_j\}$ and to the direction w is uniquely determined by m on L(w,0) and therefore is uniquely determined by the functions $\tilde{\alpha}$ and $\tilde{\gamma}$ defined by (4.5). Therefore the scattering data is uniquely determined by the functions (4.1). The scattering data, in turn, determines γ_j and determines a up to left multiplication by a constant matrix. Since $a(s,w,0) = \tilde{a}(s)$ is prescribed, the proof is complete.

Remark: One can think of $V(\lambda)$ as the initial values for the function

$$V_1(\lambda,y) = e^{\lambda y\cdot J}V(\lambda,0)e^{-\lambda y\cdot J}. \tag{4.6}$$

Replacing $V(\lambda)$ in (4.3) by $V_1(\lambda,y)$ gives the evolution of $\tilde{m}$ to all values of $\mathbb{R}^n$ which in turn corresponds to m. This is in analogy to the standard situation in IST problems.

§5 The Forward Problem for the GSGE.

Here we assume the GSGE and consider the associated spectral problem (1.17). This problem cannot easily be transformed to a single standard form, unlike the GWE. Nevertheless we shall still associate a factorization problem of Riemann-Hilbert type with (1.17).

Once again we denote

$$U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \quad J_j = \begin{pmatrix} e_j & 0 \\ 0 & -e_j \end{pmatrix} \tag{5.1}$$

and we let # denote the automorphism

$$E^{\#} = U_2^{-1}\begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} U_2 E U_2^{-1}\begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} U_2 \tag{5.2}$$

where

$$u = \operatorname{diag}(+1,-1,-1,\ldots-1)\varepsilon M_n. \tag{5.3}$$

In particular,

$$J_1^{\#} = J_1, \quad J_j^{\#} = -J_j, \ 1 < j \leqslant n. \tag{5.4}$$

We set

$$\tilde{\psi}(x,z) = U_2^{-1}\,\psi(x,z) \tag{5.5}$$

so that the spectral problem (1.17) becomes

$$\frac{\partial\tilde{\psi}}{\partial x_j} = \frac{1}{2}z\tilde{A}_j\tilde{\psi} + \frac{1}{2z}\tilde{B}_j\tilde{\psi} + \tilde{C}_j\tilde{\psi}, \tag{5.6}$$

with

$$\tilde{A}_j = U_2^{-1}A_jU_2,\ \tilde{B}_j = U_2^{-1}B_jU_2,\ \tilde{C}_j = U_2^{-1}C_jU_2 \tag{5.7}$$

The trivial (unperturbed) solution $a \equiv I$, $\gamma_j \equiv 0$ of the GSGE has the associated equation

$$\frac{\partial\tilde{\psi}}{\partial x_j} = \frac{1}{2}(zJ_j + \frac{1}{z}J_j^{\#})\tilde{\psi} \equiv J_j(z)\tilde{\psi} \tag{5.8}$$

which has a solution $\exp(x \cdot J(z))$. We view (5.6) as a perturbation of (5.8) and look for a solution in the form

$$\tilde{\psi}(x,z) = m(x,z)e^{x \cdot J(z)} \tag{5.9}$$

The equations for m are then

$$\frac{\partial m}{\partial x_j} = \frac{1}{2}\, z[\tilde{A}_jm - mJ_j] + \frac{1}{2z}\,[\tilde{B}_jm - mJ_j^{\#}] + \tilde{C}_jm. \tag{5.10}$$

As before, we normalize by

$$m(\cdot,z) \text{ is bounded.} \tag{5.11}$$

<u>Definition (5.1)</u>.

The direction w in $\mathbb{R}^n$ is *principal* if $|w_1|>|w_j|$ for $1 < j \leq n$.

Anticipating the argument below, let us consider

$$J_w(z) \equiv \Sigma w_jJ_j(z) = w_1\delta(z)J_1 + \sum_{j=2}^{n} w_j\lambda(z)J_j. \tag{5.12}$$

This matrix is diagonal with entries $\pm w_1\delta(z)$, $\pm w_j\delta(z)$, $\pm w_j\lambda(z)$, $1< j \leq n$. The set of z in $\mathbb{C}$ such that two distinct diagonal entries have the same real part always contains the set

$$\Sigma = i\mathbb{R} \cup \{z : |z| = 1\}, \tag{5.13}$$

i.e. the union of the imaginary axis and the unit circle. It is equal to this set precisely when the direction w is oblique and principal.

Definition (5.2).

The data $\{a,\alpha_j,\gamma_j\}$, where again $\alpha_j = -a^t \frac{\partial a}{\partial x_j}$, is *small in the direction* w if for every $y \perp w$,

$$\int_{-\infty}^{\infty} \| Q(s,w,y) \| ds + \frac{1}{2}\int_{-\infty}^{\infty} \|a(y+sw) - 1\| ds \leq k < 1. \tag{5.14}$$

Here again $Q = \Sigma w_j Q_j = \Sigma w_j \begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix}$. We say the data $\{a,\alpha_j,\gamma_j\}$ is *asymptotically flat in the direction* w if $\{\alpha_j,\gamma_j\}$ is asymptotically flat in the direction w.

Theorem (5.1).

Suppose the data $\{a,\alpha_j,\gamma_j\}$ is small and asymptotically flat in some principal oblique direction w. Then for each $z \in \mathbb{C} \setminus \Sigma$ there is a unique $m(\cdot,z)$ which satisfies the system (5.10), (5.11) and such that for each $y \perp w$

$$\lim_{s\to-\infty} m(y+sw,z) = I. \tag{5.15}$$

Moreover, m is bounded, $m(x,\cdot)$ is holomorphic on $\mathbb{C}\setminus\Sigma$, and $m(x,\cdot)$ has continuous limits on Σ from each of the five components of $\mathbb{C}\setminus\Sigma$.

To be specific, let us denote by m_+ the limit on Σ from the components $\{|z|>1, \mathrm{Re}z>0\}$ and $\{|z|<1, \mathrm{Re}z<0\}$ and denote by m_- the limits from the other two components.

Corollary (5.2).

There is a matrix-valued function $V:\Sigma\setminus\{\pm i\}\to SL(2n,\mathbb{C})$ such that

$$m_+(x,z) = m_-(x,z)e^{x\cdot J(z)}V(z)e^{-x\cdot J(z)} \tag{5.16}$$

As before, we define V to be the scattering data associated to (a,γ_j) and the direction w. To prove Theorem (5.1), we make two transformations. First, let

$$m'(x,z) = U_2^{-1}\begin{pmatrix} a^t & 0 \\ 0 & I \end{pmatrix} U_2 m(x,z)$$

$$= U^{-1}U_2 m(x,z). \tag{5.17}$$

Then the system (5.10) becomes

$$\frac{\partial m'}{\partial x_j} = [J_j(z),m'] + Q_j'm', \tag{5.18}$$

where

$$Q_j'(x,z) = U_2^{-1}\begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2 + \frac{1}{2z}\,[U^{-1}B_jU - J_j^{\#}]. \tag{5.19}$$

Along a line $L(w,y)$, (5.18) leads to

$$\frac{\partial\tilde{m}}{\partial s} = [J_w(z),\tilde{m}] + Q'\tilde{m},$$

$$\tilde{m}(\cdot,z) \text{ bounded, } \lim_{s\to-\infty}\tilde{m}(s,z) = I, \tag{5.20}$$

where

$$Q'(s,z) = Q'(s,z;w,y) = \Sigma w_j Q_j'(y+sw,z) \tag{5.21}$$

Although this problem is not identical to that considered in [15], nevertheless the methods of [15] apply to give existence of a unique solution $\tilde{m}(\cdot,z) = \tilde{m}(\cdot,z;w,y)$ for all $z\in\mathbb{C}\setminus\Sigma$ such that

$$\int_{-\infty}^{\infty} \| Q'(s,z)\| ds < 1. \tag{5.22}$$

The integral in (5.22) is majorized by that in (5.14) when $|z| \geq 1$. Changing to

$$m'(y+sw,z) \equiv \tilde{m}(s,z;w,y) \tag{5.23}$$

and arguing as in §2, we see that $m = U_2^{-1} Um'$ has the desired properties for all $|z| \geq 1$. To obtain results for $|z| \leq 1$ we can either use a second transformation or take advantage of a symmetry. Note that

$$J_j(1/z) = J_j(z)^{\#},$$

$$\tilde{B}_j^{\#} = \tilde{A}_j, \ \tilde{A}_j^{\#} = \tilde{B}_j, \ \tilde{C}_j^{\#} = \tilde{C}_j. \tag{5.24}$$

Therefore $m(x, \frac{1}{z})^{\#}$ satisfies the conditions for $|z| \leq 1$. This completes our sketch of the proof of Theorem (6.1).

As for the GWE, one has symmetry properties in addition to (5.24), namely that J_j, $J_j^{\#}$, $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{C}_j$ are real and

$$\begin{aligned} J_j(z) &= J_j(z)^t = -J_j(z)^{\sigma}, \\ \tilde{A}_j &= \tilde{A}_j^t = -\tilde{A}_j^{\sigma}, \\ \tilde{B}_j &= \tilde{B}_j^t = -\tilde{B}_j^{\sigma}, \\ \tilde{C}_j &= -\tilde{C}_j^t = \tilde{C}_j^{\sigma}. \end{aligned} \tag{5.25}$$

Thus one has

$$m(x,-z) = [m(x,z)^{-1}]^{t} = m(x,z)^{\sigma},$$
$$m(x,\bar{z}) = \overline{m(x,z)},\ m(x,{}^{1}/z) = m(x,z)^{\#}. \tag{5.26}$$

The symmetries of V are an immediate consequence.

Theorem (5.3).

The scattering data V has the symmetry properties

$$V(-z) = V(z)^{t} = [V(z)^{-1}]^{\sigma},$$

$$V(\bar{z}) = \bar{V}(z),\ V(\tfrac{1}{z}) = (V(z)^{-1})^{\#}. \tag{5.27}$$

The analytical properties of V can also be deduced from the results of [15]. As given above, V is defined on each of the five components of $\Sigma\setminus\{\pm i\}$. We join the two unbounded components by compactifying at ∞ and set

$$\Sigma_1 = \{\ |z| = 1,\ \mathrm{Re}\ z > 0\},$$
$$\Sigma_2 = \{\ z+\bar{z} = 0,\ |z| > 1\},$$
$$\Sigma_3 = \{\ |z| = 1,\ \mathrm{Re}\ z < 0\}, \tag{5.28}$$
$$\Sigma_4 = \{\ z+\bar{z} = 0,\ |z| < 1\}.$$

For convenience, we denote restrictions by

$$V_j = V|_{\Sigma_j},\ j = 1,3,$$
$$V_j = V^{-1}|_{\Sigma_j},\ j = 2,4 \tag{5.29}$$

Theorem (5.4).

Each V_j has a smooth extension to the closure of Σ_j. Each derivative of V-I is $O(z^{N})$ as $z \to 0$ and $O(z^{-N})$ as $z \to \infty$, for each integer $n \geq 0$.

At $\pm i$ the V_j satisfy consistency conditions:

$$V_1V_2V_3V_4(\pm i) = I. \tag{5.30}$$

More generally, for each integer $N \geq 0$ there are matrix-valued polynomials p_j of degree N such that

$$V_j(z-i) = (p_j(z-i))^{-1} p_{j+1}(z-i) + O(|z-i|^{N+1}), \quad \text{as } z \to i, \tag{5.31}$$

with similar conditions at $-i$, where we take $P_5 = P_1$.

As motivation for the next section we note that the function m' in (5.18) extends to $\mathbb{C} \setminus \Sigma$ and is the solution of the Riemann-Hilbert factorization problem (5.16) which is characterized by

$$\lim_{z\to\infty} m'(x,z) = I. \tag{5.32}$$

§6 The Inverse Problem for the GSGE.

Let $V : \Sigma \to SL(2n,\mathbb{C})$ be a matrix-valued function satisfying the symmetry conditions in Theorem (5.3) and the smoothness, decay, and consistency conditions of Theorem (5.4). Suppose also that

$$\|V(\lambda) - I\| \leq k', \; \lambda \in \Sigma \tag{6.1}$$

where k' is a sufficiently small positive constant. Then by the methods of [15], for $x \in \mathbb{R}$ there is a unique function $m'(x,\cdot)$, holomorphic on $\mathbb{C}\setminus\Sigma$ with limits on Σ, such that

$$m'_+(x,z) = m'_-(x,z)e^{x\cdot J(z)}V(z)e^{-x\cdot J(z)},$$

$$\lim_{|z|\to\infty} m'(x,z) = I. \tag{6.2}$$

The function m' is smooth up to the boundary on $\mathbb{R}^n x(\mathbb{C}\setminus\Sigma)$, and

$$m'(x,z) = I + O(z^{-1}), \; |z|\to\infty, \tag{6.3}$$

$$m'(x,z) \sim \sum_{\nu=0}^{\infty} m'_\nu(x)z^\nu, \; z\to 0. \tag{6.4}$$

Moreover, in any principal oblique direction w, for $y \perp w$ and $\lambda \in \mathbb{C}\setminus\Sigma$

$$\lim_{s\to\pm\infty} (m'(y+sw,\lambda) = \Delta_\pm \tag{6.5}$$

where $\Delta_\pm$ is diagonal. The convergence in (6.5) is $O(|s|^{-N})$ for every N, and the same is true for derivatives of m'. Also, m' and its inverse are bounded functions.

In view of these properties the functions

$$\left(\frac{\partial m'}{\partial x_j} - [J_j(z),m']\right)(m')^{-1} \tag{6.6}$$

are holomorphic on $\mathbb{C}\setminus\Sigma$, continuous across Σ except at $z = 0$, bounded as ∞, and $O(1/z)$ as $z\to 0$. For any fixed x such a function is affine in z^{-1}. Therefore m' satisfies a system of equations which we can write in the form

$$\frac{\partial m'}{\partial x_j} = [J_j(z), m'] + \frac{1}{2z}(B_j' - J_j^{\#})m + C_j' m' \qquad (6.7)$$

where $B_j' = B_j'(x)$ and $C_j' = C_j'(x)$.

The asymptotic expansion (6.4) can be differentiated, and (6.7) implies in particular that

$$B_j' m_0' = m_0' J_j^{\#}. \qquad (6.8)$$

Now m_0' is asymptotically, and rapidly, diagonal in principal oblique directions, so in such directions

$$B_j' - J_j^{\#} \to 0 \;. \qquad (6.9)$$

Because of the symmetries of V and the uniqueness of m' we obtain the symmetries

$$\begin{aligned} m'(x,-z) &= [m'(x,z)^{-1}]^t = m'(x,z)^{\sigma}, \\ m'(x,\bar{z}) &= \overline{m'(x,z)}. \end{aligned} \qquad (6.10)$$

These in turn imply that B_j' and C_j' are real, while

$$\begin{aligned} B_j' &= (B_j')^t = -(B_j')^{\sigma}, \\ C_j' &= -(C_j')^t = (C_j')^{\sigma}. \end{aligned} \qquad (6.11)$$

Thus these matrices have the form

$$B_j' = U_2^{-1} \begin{pmatrix} 0 & \beta_j \\ \beta_j^t & 0 \end{pmatrix} U_2,$$

$$C_j' = U_2^{-1} \begin{pmatrix} \alpha_j & 0 \\ 0 & \gamma_j \end{pmatrix} U_2 \quad . \tag{6.12}$$

where β_j, α_j, γ_j are real and

$$\alpha_j + \alpha_j^t = 0 = \gamma_j + \gamma_j^t. \tag{6.13}$$

We can extract more information from (6.8) by exploiting the symmetries (6.10). These symmetries imply

$$m_0' = \bar{m}_0' = (m_0'^{-1})^t = m_0^{\sigma}, \tag{6.14}$$

so

$$m_0' = U^{-1}_2 \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} U_2 \tag{6.15}$$

where f and g take values in $O(n)$. Let

$$m''(x,z) = m'(x, \tfrac{1}{z})^{\#}. \tag{6.16}$$

Then $(m_0'^{-1})^{\#} m''$ satisfies (6.2), so

$$m'(x,z) = (m_0^{-1})^{\#} \, m''(x,z), \tag{6.17}$$

Thus

$$m_0' = (m_0^{-1})^{\#}, \tag{6.18}$$

so that

$$g = g^t. \tag{6.19}$$

Since also $g^2 = g^t g \equiv 1$, g has eigenvalues ± 1. Now g depends continuously on V and $g \equiv I$ when $V = I$. Thus g is symmetric with all eigenvalues +1, hence

$$g \equiv I, \tag{6.20}$$

Combining (6.8), (6.12), (6.15), and (6.20), we obtain

$$B_j = fue_j \tag{6.21}$$

Now (6.21) implies that for $j \neq k$,

$$\begin{aligned} J_j B_k &= U_2^{-1} \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} U_2, \\ B_k J_j &= U_2^{-1} \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} U_2. \end{aligned} \tag{6.22}$$

The compatibility relations for (6.7) include

$$\begin{aligned} &\frac{\partial}{\partial x_k} C_j' + C_j' C_k + \frac{1}{4}(J_j B_k' + B_j J_k) \\ &= \frac{\partial}{\partial x_j} C_k' + C_k' C_j' + \frac{1}{4}(J_k B_j' + B_k' J_j) \end{aligned} \tag{6.23}$$

In view of (6.22) and (6.12), (6.23) implies

$$\frac{\partial \alpha_j}{\partial x_k} + \alpha_j \alpha_k = \frac{\partial \alpha_k}{\partial x_j} + \alpha_k \alpha_j. \tag{6.24}$$

These are precisely the conditions for solving for a with

$$\alpha_j = -a^t \frac{\partial a}{\partial x_j}. \tag{6.25}$$

We can require that $a \to I$ as $s \to -\infty$ along a family of principal oblique lines. Then since α_j is skew symmetric (and real),

$$a : \mathbb{R}^n \to SO(n). \tag{6.26}$$

<u>Definition (6.1)</u>.

$\{a, \gamma_j\}$ is <u>inverse data</u> for the function V.

<u>Theorem (6.1)</u>.

The inverse data $\{a, \gamma_j\}$ satisfy GSGE.

<u>Proof</u>.

Let

$$U(x) = \begin{pmatrix} a(x) & 0 \\ 0 & I \end{pmatrix} U_2 \tag{6.27}$$

and set

$$\psi(x,z) = U(x)m'(x,z)e^{x \cdot J(z)}. \tag{6.28}$$

Then the equations (6.6) become

$$\frac{\partial \psi}{\partial x_j} = \frac{1}{2} z A_j \psi + \frac{1}{2z} B_j \psi + C_j \psi \tag{6.29}$$

where

$$A_j = UJ_jU^{-1} = \begin{pmatrix} 0 & ae_j \\ e_j a^t & 0 \end{pmatrix} , \tag{6.30}$$

$$B_j = UB_j'U^{-1} = \begin{pmatrix} 0 & be_j \\ e_j b^t & 0 \end{pmatrix} \tag{6.31}$$

$$C_j = UC_j' U^{-1} - \frac{\partial U}{\partial x_j} U^{-1} ,$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix} . \tag{6.32}$$

and

$$b = a\ f\ u. \tag{6.33}$$

To complete the proof we only need to prove

$$b = ua. \tag{6.34}$$

Let us write

$$E^{\tau} = \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} \quad E \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} . \tag{6.35}$$

Then we want to prove

$$A_j = B_j^{\tau} . \tag{6.36}$$

To prove (6.36) we write the compatibility conditions for (6.29) in the notation of matrix-valued differential forms. Let

$$A = \Sigma A_j dx_j ,\ B = \Sigma B_j dx_j ,\ C = \Sigma C_j dx_j . \tag{6.37}$$

The compatibility conditions are

$$\begin{aligned} &A \wedge A = 0 = B \wedge B, \\ &dA = A \wedge C + C \wedge A,\ dB = A \wedge B + B \wedge A, \\ &dC = C \wedge C + A \wedge B + B \wedge A. \end{aligned} \tag{6.38}$$

Since $C = C^{\tau} \equiv \Sigma C_j^{\tau} dx_j$ we have

$$d(A-B^{\tau}) = (A-B^{\tau})\wedge C + C\wedge(A-B^{\tau}). \tag{6.39}$$

Now

$$A_j - B_j^{\tau} = U(J_j - (B_j')^{\#})U^{-1}$$
$$= U(J_j^{\#} - B_j')^{\#}U^{-1} \qquad (6.40)$$

and we know that $J_j^{\#} - B_j$ vanishes asymptotically in certain directions. It follows from this fact and (6.39) that $A - B^{\tau} \equiv 0$.

<u>Remarks</u>.

1. As for the GWE, the data $\{\alpha_j, \gamma_j\}$ can be recovered from the asymptotics of m' as $z \to \infty$ as in (3.10). Thus the orthogonal matrix-valued function a is also determined implicitly by these asymptotics.

2. The data $\{a, \alpha_j, \gamma_j\}$ is small in every principal oblique direction if the constant k' of (6.1) is small enough, and is asymptotically flat in every principal oblique direction.

3. As for the GWE, two functions V_1 and V_2 give rise to the same inverse data if and only if

$$V_2 = (\Delta_-)^{-1} V_1 \Delta_+ \qquad (6.41)$$

where Δ is the solution of the Riemann-Hilbert factorization problem (6.2) for a diagonal matrix-valued function on Σ. In particular, V gives the trivial solution of the GSGE if and only if V is diagonal.

§7 A Well-posed Initial-Value Problem for the GSGE.

With the same conventions as in §4, one has the same conclusion:

Theorem 7.1.

Suppose w is a principal oblique direction in $\mathbb{R}^n$.
Suppose $\tilde{a} = L(x,0) \to SO(n)$ and $\tilde{\gamma} = L(w,0) \to M_n(\mathbb{R})$ are smooth mappings such that $\tilde{\alpha} = -\tilde{a}^t \frac{d\tilde{a}}{ds}$ and $\tilde{\gamma}$ are Schwartz functions $\tilde{\gamma} + \tilde{\gamma}^t \equiv 0$, and

$$\int_{-\infty}^{\infty} \| \tilde{\alpha}(s) \| \, ds < k_0$$

where k_0 is a sufficiently small positive constant. Then there is a unique small solution $\{a, \gamma_j\}$ of the GSGE such that

$$\tilde{\alpha}(s) = a(sw)$$

$$\tilde{\gamma}(s) = \Sigma w_j \gamma_j (sw). \tag{7.1}$$

The proof is the same as the proof of the analogous result for the GWE in §4, hence is omitted.

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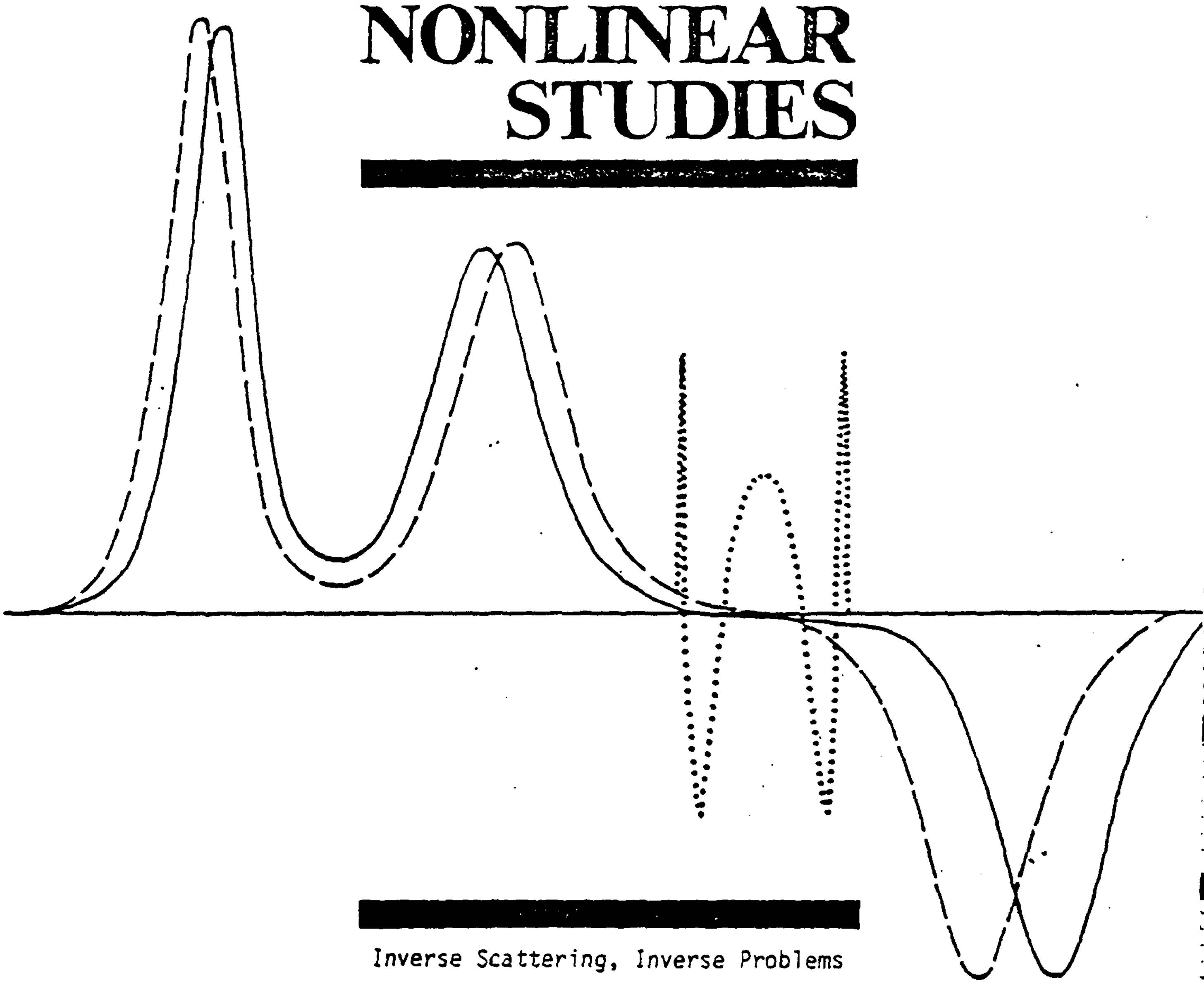

Inverse Scattering, Inverse Problems and Integrability of Nonlinear Equations in Multidimensions

by

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INVERSE SCATTERING, INVERSE PROBLEMS AND INTEGRABILITY

OF NONLINEAR EQUATIONS IN MULTIDIMENSIONS

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I. INTRODUCTION

The Inverse Scattering Transform (IST) was discovered by Gardner, Green, Kruskal, and Miura [1] who were able to relate the celebrated Korteweg-deVries (KdV) equation in the variable $q(x_0,t)$, to the classical time-independent Schrödinger equation $\Psi_{x_0 x_0} + (q(x_p;t)\Psi + k^2) = 0$. The next eigenvalue problem to receive considerable attention in this field was the so-called AKNS [2], [3] scattering problem: $\Psi_{x_0} = iKJ\Psi + q\Psi$, where J is a 2 x 2 constant real diagonal matrix and $q(k_0;t)$ is a 2 x 2 off-diagonal matrix containing the potentials. The AKNS problem is related to the nonlinear Schrödinger, modified KdV and sine-Gordon equations. The 3 x 3 extension of the AKNS problem (3 x 3 AKNS) was studied in [4] and is related to the 3-wave interaction equation in 1-spatial dimension. The N x N AKNS [5] has been studied by Shabat [6] and then by a number of authors [7] and is related to N-wave interactions.

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The IST method can be summarized as follows: The solution of the initial value problem of certain nonlinear evolution equations is essentially equivalent to solving the inverse scattering (i.e. reconstructing the potential $q(x_0)$ from appropriate scattering data) of related eigenvalue problems. The Schrödinger eigenvalue problem (with its Sturm-Liouville extension [8], [9]) and the N x N AKNS are, in our opinion, the main differential problems which have been used in connection with the IST in 1-spatial dimension. There exist several variants of the above problems [10], [11], which however should be solvable by some simple variation of the procedure used to solve the above two fundamental ones. It is therefore natural to consider extensions of these two eigenvalue problems when seeking multidimensional generalizations of the IST.

The 1-spatial dimensional extensions of the above eigenvalue problems have been recently studied by Bar Yaacov and the authors: The Schrödinger eigenvalue problem can be generalized to $\sigma\Psi_{x_1} + \Psi_{x_0x_0} + (q+k^2)\Psi = 0$ and the potential $q(x_0,x_1;t)$ is related [12] to the Kadomtsev-Petviashvili (K-P) equations [13]. There exist two important cases $\sigma = i$ and $\sigma = -1$, corresponding to KPI and KPII; their inverse problems were linearized via a Riemann-Hilbert (RH) [14] and a $\bar{\partial}$ problem [15] respectively (a $\bar{\partial}$ problem is a natural generalization of a RH problem). The N x N AKNS problem can be generalized to $\Psi_{x_0} = \sigma J_1\Psi_{x_1} + q\Psi$ [15], where J_1 is an N x N constant real diagonal matrix and $q(x_0,x_1;t)$ is an N x N off-diagonal matrix. There exist two important cases, hyperbolic ($\sigma = -1$) and elliptic ($\sigma = i$); their inverse problems were also linearized via a RH [16] and a $\bar{\partial}$ problem [17], [18]. The hyperbolic problem can be used to solve the initial value problem of the following nonlinear equation in 2-spatial and 1-temporal dimensions: the N-wave interactions, modified

KPI, and Davey-Stewartson (DS) I equation [19]. The elliptic problem can be used to solve the modified KPII and DSII. Prior to our work, interesting results regarding the solution of initial value problems in multi-dimensions can be found in [20], [21], [22].

In dealing with the above 2-spatial dimensional problems it became clear that one had to generalize the notion of the inverse scattering in general and of inverse scattering data in particular. This is also true for scalar operators as well. Namely, in both the elliptic and hyperbolic cases one can solve the inverse problem in terms of certain data $T(k_1,k_2)$, k_1, $k_2 \in R$ which we call <u>inverse data</u>. These data can be related to scattering data only in the hyperbolic case (see section 2.B). However, the elliptic case is still physically important since, although one apparently can not define physically meaningful scattering, one may still use the above formalism to solve physically interesting nonlinear evolution equations (modified KPII and DSII).

In this paper we shall consider extensions of the N x N AKNS problem to greater than 2-spatial dimensions, i.e. we shall study

$$\Psi_{x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \Psi_{x_\ell} = q\Psi, \qquad \sigma = \sigma_R + i\sigma_I$$

where $q(x_0,x)$ is an N x N matrix-valued off-diagonal function in R^{n+1} and J_ℓ are constant real diagonal N x N matrices (we denote the diagonal entries of J_ℓ by $J_\ell^1,\dots,J_\ell^N$). Alternatively, using the transformation

$$\Psi(x_0,x.k) = \mu(x_0,x,k)\exp[i(kx-\sigma x_0 kJ)], \qquad k \in R^n,$$

hwere $kx - \sigma x_0 J = \sum_{\ell=1}^{n} k_\ell(x_\ell - x_0 J_\ell)$, we shall consider

$$\mu_{x_0} + \sigma \sum_{\ell=1}^{n} (J_\ell \mu_{x_\ell} + ik_\ell[J_\ell,\mu]) = q\mu.$$

For the sake of completness only, we state the following, regarding the extensions of the Schrödinger equation in greater than 2-spatial dimensions: i) Such extensions are not known to be related to any nonlinear equations. ii) The inverse scattering of the classical 3-spatial dimensional Schrödinger equation has been studied in [23] and [24] and more recently in [25] and [26].

The system (1.3) is interesting for the following reasons:

(a) In the hyperbolic case, i.e. $\sigma = -1$ one may resolve the physically important question of inverse scattering: Given the scattering amplitude function $S(\lambda,k)$, λ, $k \in R^n$, find the potential $q(k_0,x)$.

(b) A special subcase of the hyperbolic case, namely if the J_ℓ's are constraint via

$$\frac{J_p^\ell - J_p^j}{J_r^\ell - J_r^j} = \frac{J_p^i - J_p^j}{J_r^i - J_r^j}, \qquad p,\ r = 1,\dots,n, \quad i,j,\ell=1,\dots,N$$

contains the N-wave interaction in n+1-spatial and 1-temporal dimensions [5]. This equation is the <u>only</u> known nonlinear system related to an eigenvalue problem in greater than 2-spatial dimensions (for our purpose the self-dual Yang-Mills equations is not an evolution equation).

(c) For the general σ case (except $\sigma = -1$) one cannot define physically meaningful scattering and there appears not to exist any related physically interesting nonlinear systems. However, it is mathematically interesting since it provides a unified approach to multidimensional IST.

With respect to the above note: (a) In the hyperbolic case the scattering amplitude function $S(\lambda,k)$, λ, $k \in R^n$ depends on 2n parameters, while the potential $q(x_0,x)$ depends on n+1 parameters. This fact, for $n > 1$, three important implications: i) Using the Bohr's approximation

it is possible, in an elementary way, to reconstruct $q(x_0,x)$ in closed form in terms of $S(\lambda,k)$. ii) From the above reconstruction it follows that the time evolution of $q(x_0,x;t)$ is linear, hence it is impossible for $q(x_0,x;t)$ to satisfy a nonlinear evolution equation. Thus the N-wave interaction equation must be reducible to 2-spatial dimensions. iii) The scattering data must be appropriately constrained. This "characterization" of scattering data, which is a novelty of problems in greater than 2-spatial dimensions, expresses the essence of difficulty associated with inverse problems in greater than 2-spatial dimensions. (b) In the general σ case the situation is similar to the hyperbolic case: The inverse data depends on 3n-1 parameters (while $q(x_0,x)$ depends only on n+1 parameters), it is elementary to reconstruct $q(x_0,x)$ in terms of inverse data, $q(x_0,x;t)$ cannot satisfy a nonlinear evolution equation, the only interesting problem is the solution of the characterization problem.

In this paper the following results are presented:

(a) The hyperbolic multidimensional N x N AKNS problem (i.e. eq. (.) with $\sigma = -1$) is first considered in section II: i) the N-wave interaction equation, which is contained in (.) when the J_ℓ's satisfy (.), is reduced to 2-spatial dimensions via an explicit transformation of coordinates. ii) The characterization problem is solved in two ways: The first method requires that the reconstructed $q(x_0,x)$ must be independent of k. The other, explores the analytic structure of the inverse data $T_{\pm}(\lambda,k)$, λ, $k \in R^n$ with respect to k. In more details: In IIA we introduce eigenfunctions $\mu^+(x_0,x,k)$, $\mu^-(x_0,x,k)$ analytic with respect to k_1, for $k_{1_I} \geq 0$ and $k_{1_I} \leq 0$ respectively. With the aid of these eigenfunctions

we can solve the inverse problem as well as reconstruct $q(x_0,x)$ in terms of a Riemann-Hilbert (RH) problem uniquely defined in terms of the inverse data $T_{\pm}(\lambda,k)$. The relevant formulae are direct generalizations of the analogous formulae in 2-spatial dimensions [16]. These formulae provide a less effective way of reconstructing $q(x_0,x)$ than the Bohr's approximation, however they proive the basis for the solution of the characterization problem. In IIB we relate the inverse data $T_{\pm}(\lambda,k)$ to the scattering amplitude function $S(\lambda,k)$ via a linear integral equation. Also we give the Bohr's approximation reconstruction of the potential $q(x_0,x)$. In IIC we solve the characterization problem for $T_{\pm}(\lambda,k)$ using the results of IIA. Because of IIB this also provides a solution of the characterization problem for $S(\lambda,k)$. In IID we show that if the J_ℓ's are constrained via equations (.), a new $\hat{k}$ can be introduced (which is a combination of the previous k's), the scattering data depends only on two parameters, and the characterization problem is by-passed. This also provides an additional motivation to reduce the N-wave interactions to 2-spatial dimensions. In IIE we apply the direct linearizing method to the solution of the inverse problem.

(b) The general σ case is then considered in section III. The associated characterization problem was first solved in [28] via the "T equation". In this paper both the T equation and its derivation are somewhat simplified by using slightly different inverse data than those of [27], [28]. In more details: In IIIA we introduce an eigenfunction $\mu(x_0,x,k)$ bounded for all $k \in C^n$. Using a $\bar{\partial}$ problem we solve both the inverse problem as well as reconstruct $q(x_0,x)$ in terms of the inverse data

$T^{ij}(k_1,\ldots,k_n,m_2,\ldots,m_n)$, $k_i \in C$, $m_i \in R$, $i,j = 1,\ldots,N$. Similar formulae were given in [27] for $\sigma = i$, in [28] for general σ, and provide generalizations of the analogous formulae in 2-spatial dimensions [16]. Here we use a slightly different "symmetry condition" of the underlying Green's function. The above formulae, like in the hyperbolic case, provide a less effective way of reconstructing $q(x_0,x)$ than the Bohr's approximation but again provide the basis for the solution of the characterization problem. In IIIA we also give the Bohr's approximation. In IIIB we derive the T equation. Also, the reconstructed $q(x_0,x)$ appears to depend on k; furthermore there exist various inversion formulae for the solution of the inverse problem. It is explicitly shown here that the equality of the inversion formulae is equivalent to $q(x_0,x)$ being independent of k. In IIIC it is shown that if the J_ℓ's are constrained via equations (.), a new $\hat{k}$ can be introduced. T^{ij} depends only on $n+1$ parameters, and the characterization problem is by-passed. This is consistent with the fact that the T equation is identically zero in this case.

Concluding this introduction we note that all results presented here are formal: Both the direct and inverse problems involve linear integral equations. One still needs to establish existence and uniqueness of the solution of these equations. Thus, strictly speaking, "solved" should be replaced by "formally solved". However, if $q(x_0,x)$ delays sufficiently fast for large x_0,x and if its appropriate norm is sufficiently small, all equations presented here are well defined.

II. THE HYPERBOLIC SYSTEM

We first consider equations (.)-(.) when $\sigma = -1$.

A. The Inverse Problem

Let $\pi_0 f$, $\pi_+ f$, $\pi_- f$ denote the diagonal, strictly upper diagonal, and strictly lower diagonal parts of the matrix f. Let $\hat{J}_\ell f \doteq [J_\ell, f]$, for any diagonal matrix J_ℓ, thus $\exp(\hat{J}_\ell)f = \exp(J_\ell)f\exp(-J_\ell)$. If $k \in C^n$, $x \in R^n$ then $kx \doteq \sum_{\ell=1}^{n} k_\ell x_\ell$, $kJ \doteq \sum_{\ell=1}^{n} k_\ell J_\ell$. Let $\{f\}^{ij}$ denote the ij^{th} component of the matrix f and $\{f(x + x_0 J)\}^{ij} = f^{ij}(x_1 + x_0 J_1^i, \ldots, x_n + x_0 J_n^i)$. Rewrite equations (.) in such a way that $J_1^1 > J_1^2 > \ldots > J_1^N$.

Proposition 2.1.

A solution of (.) with $\sigma = -1$, bounded for all complex values of $k_1 = k_{1_R} + ik_{1_I}$ and tending to I, the unit N x N matrix, as $k_1 \to \infty$ is given by

$$\mu(x_0,x,k) = \begin{cases} \mu^+(x_0,x,k), & k_{1_I} \geq 0 \\ \mu^-(x_0,x,k), & k_{1_I} \leq 0 \end{cases} ,$$

where $\mu^\pm(x_0,x,k)$ satisfy the following linear integral equations:

$$\mu^\pm(x_0,x,k) = I + \int_{-\infty}^{x_0} d\xi_0 e^{i(x_0-\xi_0)k\hat{J}} (\pi_0 + \pi_\pm)(q\mu^\pm)(\xi_0, x+(x_0-\xi_0)J, k)$$
$$- \int_{x_0}^{\infty} d\xi_0 e^{i(x_0-\xi_0)k\hat{J}} \pi_\mp (q\mu^\pm)(\xi_0, x+(x_0-\xi_0)J, k),\quad k_1 \in C, k_2, \ldots, k_n \in R$$

or, in component form

$$\mu^{\pm ij}(x_0,x,k) = \zeta^{ij} + \int_{-\infty}^{x_0} d\xi_0 e^{i(x_0-\xi_0)\sum_{\ell=1}^{n} k_\ell (J_\ell^i - J_\ell^j)} (\pi_0 + \pi_\pm)(q\mu^\pm)^{ij}(\xi_0, x+(x_0-\xi_0)J^i, k)$$
$$- \int_{x_0}^{\infty} d\xi_0 e^{i(x_0-\xi_0)\sum_{\ell=1}^{n} k_\ell (J_\ell^i - J_\ell^j)} \pi_\mp (q\mu^\pm)^{ij}(\xi_0, x+(x_0-\xi_0)J^i, k),$$

$k_1 \in C$, $k_2, \ldots, k_n \in R$

where $\zeta^{ij} = 0$ if $i \neq j$ and $\zeta^{ij} = 1$ if $i = j$.

To derive equations (.) note that the Fourier transform, $\tilde{\Psi}_\sigma(x_0,m,k) = \int_{R_n} d\xi\, \Psi_\sigma(x_0,\xi,k)\exp(-im\xi)$, of equations (.) implies that Ψ_σ satisfies

$$\tilde{\Psi}_\sigma(x_0,x,k) = \frac{1}{(2\pi)^n}\int_{R_n} dm\, e^{imx-i\sigma x_0 mJ} A(m,k) +$$

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{x_0} d\xi_0 \int_{R_n} d\xi \int_{R_n} dm\, e^{im(x-\xi)-i\sigma(x_0-\xi_0)mJ}(q\Psi_\sigma)(\xi_0,\xi,k)$$

where $A(m,k)$ is an arbitrary function of m,k. Thus μ_σ satisfies

$$\mu_\sigma(x_0,x,k) = \frac{1}{(2\pi)^n}\int_{R_n} dm\, e^{imx-i\sigma x_0 mJ - i\sigma x_0 k\hat{J}} A(m,k)$$

$$+\frac{1}{(2\pi)^n}\int_{-\infty}^{x_0} d\xi_0 \int_{R_n} d\xi \int_{R_n} dm\, e^{im(x-\xi)-i\sigma(x_0-\xi_0)mJ-i\sigma(x_0-\xi_0)k\hat{J}}(q\mu_\sigma)(\xi_0,\xi,k).$$

Equation (.) with $\sigma = -1$ and an appropriate choice of $A(m,k)$ (see [18]) implies (.) where we have only used

$$\frac{1}{(2\pi)^n}\int_{R_n} d\xi \int_{R_n} dm\, e^{im(x-\xi)+im(x_0-\xi_0)J} f(\xi) = \int_{R_n} d\xi\,\zeta(\xi-[x+(x_0-\xi_0)J])f(\xi) =$$

$$f(x + (x_0-\xi_0)J).$$

For real values of k_1 one may relate μ^+ and μ^-:

<u>Proposition 2.2.</u>

Let $\mu^\pm$ be defined by equation (.), then

$$\mu^+(x_0,x,k) - \mu^-(x_0,x,k) = \int_{R_n} d\lambda\,\mu^-(x_0,x,\lambda)e^{i\lambda(x+x_0J)} f(\lambda,k)e^{-ik(x+x_0J)},$$

$$k,\lambda \in R^n,$$

hwere $f(\lambda,k)$ is defined via

$$f(\lambda,k) - \int_{R_n} dm\, T_+(\lambda,m)f(m,k) = T_+(\lambda,k) - T_-(\lambda,k),$$

in terms of the <u>inverse data</u>

$$T_{\pm}(\lambda,k) \doteq \frac{1}{(2\pi)^n}\int_{R_{n+1}} d\xi_0 d\xi e^{-i\lambda(\xi+\xi_0 J)}\pi_{\pm}(q\mu^{\mp})(\xi_0,\xi,k)e^{ik(\xi+\xi_0 J)}, \quad k,\lambda \in R^n.$$

The derivation of the above result is similar to that of the 2-dimensional case (see [18]) and is outlined in Appendix A.

<u>Remarks</u> 1. Equation (.) implies that the relevant integrants are analytic in k_1 for $k_{1_I} \geq 0$. Thus, assuming that (.) has no homogeneous solutions, $\mu^+(x_0,x,k)$ is a holomorphic function of k_1 for $k_{1_I} \geq 0$. Similarly for μ^-. Hence, equation (.) defines a sectionally holomorphic function of k_1 having a jump across $k_{1_I} = 0$. This jump is given by proposition 2 in terms of the inverse data $T_{\pm}$.

2. Equations (.), (.) imply that $\mu(x_0,x,k)$ is, in general, defined for complex values of k_1 but only for real values of $k_2,\ldots,k_n$. That is, we solve (.) for $k_1 \in C$, $k_2,\ldots,k_n \in R$. This is in contrast to the results of [28] where (.) is solved for $k \in C^n$. Thus in a sense we solve here a weaker problem and hence our approach is considerably simpler than that of [28]. It is interesting that both the questions of inverse scattering and of solvability of the related nonlinear equations can be resolved using eigenfunctions of only one complex variable.

3. We note the remarkable fact that equation (.) is solvable in <u>closed form</u>. This is because its kernel is strictly upper triangular. For example if $N = 2$ then $f^{22} = 0$, $f^{21} = -T_+^{12}$, $f^{11}(\lambda,k) = \int_{R_1} dm\, T_+^{12}(\lambda,m)T_-^{21}(m,k)$; similar formulae exist for any N.

Proposition 2.3.

The potential $q(x_0,x)$ of equation (.) with $\sigma = -1$, can be reconstructed from

$$q(x_0,x) = -\frac{1}{2\pi}[J_1,\int_{R_{n+1}} d\hat{k}_1 d\lambda\mu^-(x_0,x,\lambda)e^{i\lambda(x+x_0J)} f(\lambda,\hat{k}_1,k_2,\ldots,k_n)$$

$$\times\, e^{-i\lambda k(x+x_0-J)-i[(\hat{k}_1-k_1)x_1 + x_0(\hat{k}_1-k_1)J_1]}],$$

where μ^- can be obtained from

$$\mu^-(x_0,x,k) + \frac{1}{2\pi i}\int_{R_1} d\hat{k}_1$$

$$\times\, \frac{\int_{R_n} d\lambda\mu^-(x_0,x,\lambda)e^{i\lambda(x+x_0J)} f(\lambda,\hat{k}_1,k_2,\ldots,k_n)e^{-ik(x+x_0J)-i[(\hat{k}_1-k_1)x_1+(\hat{k}_1-k_1)x_0J_1]}}{\hat{k}_1 - k + i0} = I.$$

The function $f(\lambda,k)$ is defined by (.) in terms of the inverse data $T_{\pm}(\lambda,k)$.

To derive the above note that (.) defines a nonlocal RH problem in the complex k_1-plane for the sectionally holomorphic matrix function $\mu(x_0,x,k)$. By taking its "minus" projection [30] it follows that $\mu^-(x_0,x,k)$ solves (.). Also if one seeks an asymptotic expansion of $\mu(x_0,x,k)$ for large k_1 in the form $\mu(x_0,x,k) = I + \mu_1(x_0,x,k_2,\ldots,k_n)/k_1 + O(1/k_1^2)$ one obtains from (.) $q = -i\hat{J}\mu_1$. This and large k_1 asymptotics of (.) implies (.).

B. Inverse Data and Scattering Amplitude Function

We now find a relationship between the inverse data $T_{\pm}(k,\lambda)$, $k,\lambda \in R^n$ and the classical scattering amplitude function $S(k,\lambda)$. $T_{\pm}$ are defined in terms of $\mu^{\pm}$, S is defined below: Let $\phi(x_0,x)$ be the general solution of (.) such that $\phi(x_0,x) \to F(x+x_0J)$ as $x_0 \to -\infty$ (it follows from (.) with $\sigma = -1$ that for large x_0, sine $q \to 0$, ϕ becomes a function of $x+x_0J$ only). Then, by definition, the scattering operator $\tilde{S}$ is given by $G = \tilde{S}F$, where $G(x+x_0J)$ is the value of Φ as $x_0 \to +\infty$. Equation (.) implies that Φ solves

$$\Phi(x_0,x) = F(x+x_0J) + \frac{1}{(2\pi)^n}\int_{R_n} dm\, e^{im(x+x_0J)}\int_{-\infty}^{x_0} d\xi_0\int_{R_n} d\xi e^{-im(\xi+\xi_0J)}(q\Phi)(\xi_0,\xi)$$

Let $F(x+x_0J) = 1/(2\pi)^n \int_{R_n} dk \exp[ik(x+x_0J)]\hat{F}(k)$, hence

$\Phi(x_0,x) = 1/(2\pi)^n \int_{R_n} dk\, \Psi_L(x_0,x,k)\hat{F}(k)$, where Ψ_L solves

$$\Psi_L(x_0,x,k) = e^{ik(x+x_0J)} + \frac{1}{(2\pi)^n}\int_{R_n} dm\, e^{im(x+x_0J)}\int_{-\infty}^{x_0} d\xi_0\int_{R_n} d\xi e^{-im(\xi+\xi_0J)}(q\Psi_L)(\xi_0,\xi,k).$$

Letting $x_0 \to +\infty$, the left hand side of (.) becomes $\hat{S}F$ and the right hand side of (.) involves the Fourier transform of $\int_{R_{n+1}} d\xi_0 d\xi \exp[-im(\xi+\xi_0J)](q\Phi)(\xi_0,\xi)$. Thus, the Fourier transform of $\tilde{S}F$ is given by:

$$(\tilde{S}F)(m) \doteq \hat{F}(m) + \frac{1}{(2\pi)^n}\int_{R_n} dk\int_{R_{n+1}} d\xi_0 d\xi\, e^{-im(\xi+\xi_0J)}(q\Psi_L)(\xi_0,\xi,k)\hat{F}(k).$$

<u>Definition 2.1.</u>

The Fourier transform of the scattering operator $\tilde{S}$ is uniquely defined in terms of $S(m,k)$ where

$$S(m,k) \doteq \frac{1}{(2\pi)^n} \int_{R_{n+1}} d\xi_0 d\xi_1 e^{-im(\xi+\xi_0 J)} (q\Psi_L)(\xi_0,\xi,k)$$

$$= \frac{1}{(2\pi)^n} \int_{R_{n+1}} d\xi_0 d\xi_1 e^{-im(\xi+\xi_0 J)} (q\mu_L)(\xi_0,\xi,k) e^{ik(\xi+\xi_0 J)}, \quad m,k \in R^n$$

and Ψ_L is defined by (.), while $\mu_L = \Psi_L \exp[-ik(x+x_0 J)]$ is defined by (using (.) and (.))

$$\mu_L(x_0,x,k) = I + \int_{-\infty}^{x_0} d\xi_0 e^{i(x_0-\xi_0)k\hat{J}} (q\mu_L)(\xi_0, x+(x_0-\xi_0)J, k),$$

$$k \in R^n.$$

<u>Proposition 2.4.</u>

The eigenfunctions $\mu_{\pm}$, used to define $T_{\pm}$, and the eigenfunction μ_L used to define S are related via:

$$\mu^-(x_0,x,k) - \mu_L(x_0,x,k) = -\int_{R_n} dm\ \mu_L(x_0,x,m) e^{im(x+x_0 J)} T_+(m,k) e^{-ik(x+x_0 J)}$$

$$= \int_{R_n} dm\ \mu^-(x_0,x,m) e^{im(x+x_0 J)} \Lambda(m,k) e^{-ik(x+x_0 J)}, \quad k,m \in R^n,$$

where $\Lambda(m,k)$ is expressed in terms of $T_+(m,k)$:

$$\Lambda(m,k) - \int_{R_n} d\tau\ T_+(m,\tau)\Lambda(\tau,k) = -T_+(m,k).$$

If one studies the steps involved in establishing (.) and (.) the above relationships follow by inspection.

<u>Remarks</u>.

I. Equation (.) can be solved in closed form. For example in the 2 x 2 case $\Lambda = -T_+$, in the 3 x 3 case $\Lambda(m,k) = -T_+(m,k) - \tilde{\Lambda}(m,k)$, where

all entries of $\tilde{\Lambda}(m,k)$ are zero except $\{\tilde{\Lambda}(m,k)\}^{13} = \int_{R_n} d\tau \{T_+(m,\tau)\}^{12}\{T_+(\tau,k\}^{23}$.

2. Using (.), one may verify that the second and third equations of (.) are equal (see Appendix B).

3. Given μ^-, equation (.) yields T_+, equation (.) yields Λ, and equation (.) yields μ_L.

<u>Proposition 2.5.</u>

Let $M(\lambda,k)$ be defined by

$$M(\lambda,k) \doteq \frac{1}{(2\pi)^n}\int_{R_{n+1}} d\xi_0 d\xi\, e^{-i\lambda(\xi+\xi_0 J)}(q\mu^-)(\xi_0,\xi,k)e^{ik(\xi+\xi_0 J)},$$

$$\lambda,k \in R^n$$

i.e. $T_+ = \pi_+ M$. Then a) $S(\lambda,k)$ is given in closed form in terms of M:

$$S(\lambda,k) = M(\lambda,k) - \int_{R_n} dm\, M(\lambda,k)\Lambda(m,k), \qquad \lambda,k \in R^n$$

where $\Lambda(m,k)$ is defined in terms of $\pi_+ M$ by (.).

b) $T_{\pm}$ are expressed via linear integral equations in terms of S:

$$T_{\pm}(\lambda,k) + \pi_{\pm}\int_{R_N} dm\, S(\lambda,m)T_{\pm}(m,k) = \pi_{\pm}S(\lambda,k), \qquad \lambda,k \in R^n.$$

To derive the above results first multiply equation (.) by $1/(2\pi)^n \exp[-i\lambda(x+x_0 J]q(x_0,x)$ from the left and by $\exp[ik(x+x_0 J)]$ from the right and integrate over R_n to obtain

$$M(\lambda,k) - S(\lambda,k) = -\int_{R_n} dm\, S(\lambda,m)T_+(m,k) = \int_{R_n} dm\, M(\lambda,m)\Lambda(m,k), \quad \lambda,k \in R^n$$

the π_+ projection of the first equation of (.) yields (.)$^+$. Also to equations (.), (.), (.), (.) corresponds analogous ones

for μ^+, T_-, P (where $P(\lambda,k)$ is defined like (.) in terms of μ^+). To obtain these equations let $\mu^- \to \mu^+$, $T_+ \to T^-$, $M \to P$. The $_-$ projection of the equation corresponding to (.) yields $(.)^-$.

C. The Characterization Problem.

The potential $q(x_0,x)$ depends on n+1 parameters, while the inverse data (as well as the scattering amplitude function) depend on 2n parameters. Thus unless $n+1 = 2n$, i.e. $n=1$ (the 2-spatial dimensional case) the inverse data must be appropriately constrained.

Equation (.) implies that the right hand side of (.) will in general depend on $k_2,\ldots,k_n$ unless $f(\lambda,k)$ is appropriately constrained. This provides the first method of solving the characterization problem of the inverse data: Choose $f(\lambda,k)$ so that the reconstructed $q(x_0,x)$ is independent of $k_2,\ldots,k_n$. This corresponds to the well known "miracle condition" of Newton [24] in the inverse scattering of the classical 3-dimensional Schrödinger equation. However, it has the disadvantage that it involves $\mu^-(x_0,x,k)$ which depends via (.) on $f(\lambda,k)$.

More explicit constraints on $f(\lambda,k)$ can be found by utilizing the fact that f is defined in closed form in terms of $T_\pm$, which are defined in terms of analytic eigenfunctions. Actually $T_\pm$ satisfy the following "analyticity" constraints:

Proposition 2.6.

Let

$$E_\pm(x_0,x,k) \doteq \int_{R_n} d\lambda\, e^{i\lambda(x+x_0J)} T_\pm(\lambda,k) e^{-ik(x+x_0J)} .$$

Then $E_\pm$ satisfy

$$E_\pm(x_0,x,k) = \int_{R_1} d\xi_0 \pi_\pm(q\mu^\mp)(\xi_0, x+(x_0-\xi_0)J,k),$$

i.e. the functions E_+, E_- are analytic functions in the lower and upper halves of the k_1-complex plane respectively.

To derive (.) multiply (.) by $\exp[i\lambda(x+x_0 J)]$ from the left, by $\exp[-ik(x+x_0 J)]$ from the right and integrate over R_n. Since μ^- is analytic in the lower half k_1-plane, so is E_+; similarly for E_-.

Remarks.

I. The above analyticity constraint is conceptually analogous to the Faddeev condition [23] in the inverse scattering of the classical 3-dimensional Schrödinger equation.

2. Comparing the above method of solving the characterization problem to that used in [28] we note: In [28] the inverse data are defined in terms of an eigenfunction μ of n complex variables $k_1,\ldots,k_n$. This eigenfunction is not analytic with respect to any k_i, i.e. $\partial\mu/\partial\bar{k}_i \neq 0$, and the characterization problem is solved by utilizing the symmetry of $\partial\mu/\partial\bar{k}_i\partial\bar{k}_j$ with respect to i, j. Here we work with eigenfunctions which are not bounded for complex $k_2,\ldots,k_n$ but which are analytic with respect to k_1, hence the characterization problem is solved by utilizing precisely this analyticity.

D. The N-Wave Interaction Equations Are 2-Dimensional.

The N-wave interaction equations for potentials with components $q^{ij}(x_0,x,t)$ are given by

$$q_t^{ij} = -\alpha_{ij}q_{x_0}^{ij} + \sum_{\ell=1}^{n}(\alpha_{ij}J_\ell^i - B_\ell^i)q_{x_\ell}^{ij} - \sum_{\ell=1}^{n}(\alpha_{i_\ell} - \alpha_{\ell_j})q^{i\ell}q^{\ell j}.$$

Equations (.) are the compatibility conditions of (.), in the special case that the J_ℓ's satisfy equations (.), and of

$$\mu_t + \sum_{\ell=1}^{n} B_\ell\mu_{x_\ell} + i\sum_{\ell=1}^{n} k_\ell[B_\ell,\mu] = A\mu,$$

where A and B are given by $A^{ij} = -\alpha_{ij}q^{ij}$, $\alpha_{ij} = (B_\ell^i - B_\ell^j)/(J_\ell^i - J_\ell^j)$.

Hence, the formalism derived in the previous sections can be used to linearize (.), also the time evolution of the inverse data $f(\lambda,k)$ is given by

$$\frac{\partial f(\lambda,k;t)}{\partial t} = i \sum_{\ell=1}^{n} (f(\lambda,k;t)k_\ell B_\ell - B_\ell f(\lambda,k;t)).$$

(To derive (.) use (.) in a similar way to that used in the 2-dimensional case, see [30]). However, because of the constraint (.) the above formalism can be simplified.

Equations (.) can be used in two interrelated ways:

i) From both equations (.) and (.) it follows that one may introduce a new parameter $\hat{k}$, which is a combination of $k_1,\ldots,k_n$ iff

$$\sum_1^n (J_\ell^i - J_\ell^j)k_\ell = (J_1^i - J_1^j)\hat{k}, \quad \text{for all} \quad i,j = 1,\ldots,n.$$

It is interesting that if equations (.) are valid, then equations (.) are always solvable for $\hat{k}$. This fact will be illustrated later for the general σ case. Here we only point out that if N=2 equations (.) are always solvable. Hence, the inverse problem for N=2 in n+1 spatial dimensions can always be solved using only one complex variable.

ii) With the introduction of a new $\hat{k}$, the inverse data depends only on 2 parameters. This suggests that if the J_ℓ's satisfy (.) then equations (.), (.) are reducible to 2-spatial dimensions. This is indeed the case:

Proposition 2.7.

Let $\bar{\bar{J}}$ denote the matrix formed from the J_ℓ's, i.e.

$$\bar{\bar{J}} = \begin{pmatrix} J_1^1 & J_2^1 & \dots & J_p^1 & \dots & J_n^1 \\ \vdots & & & & & \\ J_1^\ell & J_2^\ell & \dots & J_p^\ell & \dots & J_n^\ell \\ \vdots & & & & & \\ J_1^N & J_2^N & \dots & J_p^N & \dots & J_n^N \end{pmatrix} \doteq (\vec{J}_1\ \vec{J}_2\ \dots\ \vec{J}_p\ \dots\ \vec{J}_n) \doteq \begin{pmatrix} \vec{J}^1 \\ \vdots \\ \vec{J} \\ \vdots \\ \vec{J}^N \end{pmatrix}$$

If equations (.) are valid then

$$\vec{J}^\ell = \vec{J}^2 + \alpha^\ell(\vec{J}^1 - \vec{J}^2), \quad \vec{J}_p = \alpha_p\vec{J}_1 + b_p\vec{J}_2, \ \alpha,\ a,\ b \text{ constants.}$$

Using equations (.), equations (.) with $\sigma = -1$ become

$$L^1\Psi^{1j} = (q\Psi)^{1j}$$

$$L^2\Psi^{2j} = (q\Psi^{2j})$$

$$\{\alpha^k L^1 + (1-\alpha^k)L^2\}\Psi^{kj} = (q\Psi)^{kj}, \quad k \geq 3, \quad j = 1,2,\dots,N,$$

where

$$L^i \doteq \frac{\partial}{\partial x_0} - J_1^i \frac{\partial}{\partial \chi_1} - J_2^i \frac{\partial}{\partial \chi_2}, \quad i=1,2; \quad \frac{\partial}{\partial \chi_1} \doteq \sum_{\substack{\ell=1 \\ \ell\neq 2}}^{n} a_\ell \frac{\partial}{\partial x_\ell}, \quad \frac{\partial}{\partial \chi_2} \doteq \sum_{\ell=1}^{n} b_\ell \frac{\partial}{\partial x_\ell} .$$

The transformation

$$\xi_0 = x_0, \quad \xi_1 = \chi_1, \quad \xi_2 = \chi_2 + \frac{J_1^1 J_2^2 - J_1^2 J_2^1}{J_1^1 = J_1^2} x_0$$

yields

$$L^i = \frac{\partial}{\partial \xi_0} - J_1^i\left(\frac{\partial}{\partial \xi_1} + R\frac{\partial}{\partial \xi_2}\right), \quad i = 1,2; \quad R \doteq \frac{J_2^2 - J_2^1}{J_1^2 - J_1^1} .$$

Thus

$$L^i = \frac{\partial}{\partial \xi_0} - J_1^i \frac{\partial}{\partial \tau}\text{, on the characteristic coordinate } \tau : \frac{d\xi_2}{d\xi_1} = R.$$

To derive the above results first note that (.) imply that $(J_p^\ell - J_p^2)/(J_p^1 - J_p^2) = (J_r^\ell - J_r^2)/(J_r^1 - J_r^2) = \alpha^\ell$, i.e. $J_p^\ell = \alpha^\ell(J_p^1 - J_p^2) + J_p^2$, which is the component version of (. a). Hence there are two independent row vectors $\vec{J}^1$, $\vec{J}^2$, *which implies that these exist at most two* independent column vectors, say $\vec{J}_1$, $\vec{J}_2$, and (. b) is valid. Thus (.) becomes

$$\frac{\partial\Psi}{\partial x_0} = J_1 \sum_{\substack{\ell=1 \\ \ell\neq 2}}^{n} \alpha_\ell \frac{\partial\Psi}{\partial x_\ell} + J_2 \sum_{\ell=2}^{n} b_\ell \frac{\partial\Psi}{\partial x_\ell} + q\Psi.$$

Introducing the coordinates χ_1, χ_2 and then writing (.) in component form we obtain

$$\frac{\partial\Psi^{kj}}{\partial x_0} = J_1^k \frac{\partial\Psi^{kj}}{\partial \chi_1} + J_2^k \frac{\partial\Psi^{kj}}{\partial \chi_2} + (q\Psi)^{kj} .$$

Equations (.) follow from (.), where for $k \geq 3$ we use $J_p^k = J_p^2 + \alpha^k(J_p^1 - J_p^2)$, $p = 1,2$. Thus there exists only two important operators L^i, $i = 1,2$. Using $x_0 = \xi_0$, $\xi_1 = \chi_1$, $\xi_2 = \chi_2 + \beta x_0$ these operators become $\partial/\partial\xi_0 - J_1^i\, \partial/\partial\xi_1 + (\beta - J_2^i)\, \partial/\partial\xi_2$. For the existence of a characteristic coordinate τ we require

$$\frac{\beta - J_2^1}{J_1^1} = \frac{\beta - J_2^2}{J_1^2} = R .$$

The first equation above determines β (see (.)), the second determines R (see (.)).

Manakov [31] also suggested that the N-wave interactions are reducible (see also [32]).

E. Direct Linearization

The essence of the "direct linearizing method" [33] is the existence of certain linear integral equations (such as (.)), the solution of which are related via some formulae (such as (.)) to the solution of certain linear eigenvalue problems (such as (.)). Clearly, the above formalism provides a formal solution of the inverse problem. Also, if there exist nonlinear evolution equations related to the underlying linear eigenvalue problems, such a formalism provides also a formal linearization of the nonlinear equations. Further discussion of the above method for one and two spatial dimensional problems can be found in [34]. Many applications can be found in [35] and [36]. Here, we only point out that, although the direct linearizing method is both straightforward and effective in producing special solutions, it is not suitable for solving initial value problems. This is because, given $q(x_0,x,t=0)$ it is not clear how the measure-contour-inverse data can be chosen (see below).

Proposition 2.8.

Let $u(x_0,x,k)$ be a solution of the linear integral equation in k_1

$$\mu(x_0,x,k) + \frac{1}{2\pi i}\int_C d\sigma(\lambda,\hat{k}_1)\int_{R_n}$$

$$\times \frac{d\lambda\, u(x_0,x,\lambda)e^{i\lambda(x+x_0J)}f(\lambda,\hat{k}_1,k_2,\ldots,k_n)e^{-ik(x+x_0J)-i[(\hat{k}_1-k_1)x+(\hat{k}_1-k_1)x_0J]}}{\hat{k}_1-k} = I,$$

where the measure $d\sigma(\lambda,\hat{k}_1)$, contour C, matrix $f(\lambda,k)$ are essentially arbitrary. Assume that the homogeneous integral equation corresponding

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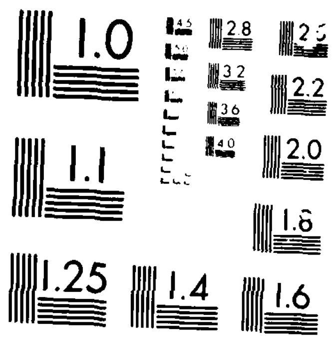
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.8
3.2
3.6
4.0

to (.) has only the zero solution. Then

$$q(x_0,x) \doteq -\frac{1}{2\pi}\hat{J}\int_C d\rho(\lambda,\hat{k}_1)\int_{R_n} d\lambda\mu(x_0,x,\lambda)e^{i\lambda(x+x_0J)}$$

$$\times f(\lambda,\hat{k}_1,k_2,\ldots,k_n)e^{-ik(x+x_0J)-i[(\hat{k}_1-k_1)x+(\hat{k}_1-k_1)x_0J]}$$

solves equation (.).

To derive the above results, define the linear operators L_k, $P_{x_0,x,f}$ via

$$(L_k g)(x_0,x) \doteq \frac{\partial}{\partial x_0} g - \sum_{\ell=1}^{n}(J_\ell g_{x_\ell} + ik_\ell[J_\ell,g]),$$

$$(P_{x_0,x,f}\, g)(k_1) \doteq -\frac{1}{2\pi i}\int_C d\rho(\lambda,\hat{k}_1)\int_{R_n}$$

$$\times \frac{d\lambda g(x_0,x,\lambda)e^{i\lambda(x+x_0J)} f(\lambda,\hat{k}_1,k_2,\ldots,k_n)e^{-ik(x+x_0J)-i[(\hat{k}_1-k_1)x+(\hat{k}_1-k_1)x_0J]}}{k_1 - k_1}$$

By direct computation one may verify that

$$[L_k,P_{x_0,x,f}]g(x_0,x,k) = -\frac{1}{2\pi}\hat{J}\int_C d\rho(\lambda,\hat{k}_1)\int_{R_n} d\lambda g(x_0,x,\lambda)e^{i\lambda(x+x_0J)}$$

$$\times f(\lambda,\hat{k}_1,k_2,\ldots,k_n)e^{-ik(x+x_0J)-i[(\hat{k}_1-k_1)x+(\hat{k}_1-k_1)x_0J]} .$$

Equation (.) can be written as

$$\mu(x_0,x,k) = I + (P_{x_0,x,f}\mu)(k_1).$$

Applying the operator $L_k - qI$ on (.) and using (.) it follows that

$$(L_k-q)\mu(x_0,x,k) = P_{x_0,x,f}\{(L_k-q)\mu(x_0,x,k)\}.$$

Hence, assuming that (.) has no nontrivial homogeneous solutions the above equation implies $(L_k-q)\mu = 0$ which is equation (.).

III. THE GENERAL σ CASE

We now consider equations (.)-(.) for arbitrary complex σ.

A. The Inverse Problem

Proposition 3.1.

The solution of (.), bounded for all complex values of k and tending to I for large k is given by

$$\mu^{ij}(x_0,x,k) = \zeta^{ij} + \operatorname{sgn}\frac{(\sigma_I J_1^i)}{2\pi i}\int_{R_2} d\xi_0 d\xi_1 \frac{e^{i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,k)}}{(x_1-\xi_1)-\sigma J_1^i(x_0-\xi_0)}$$

$$\times (q\mu)^{ij}\left(\xi_0,\xi_1,x_2-(x_1-\xi_1)\frac{J_2^i}{J_1^i},\ldots,x_n-(x_1-\xi_1)\frac{J_n^i}{J_1^i},k\right),\quad k\in C^n,$$

where β^{ij} is defined by

$$\beta^{ij}(x_0,x_1,k) \doteq \sum_{\ell=1}^{n}\frac{J_\ell^i - J_\ell^j}{\sigma_I}\left[x_0|\sigma|^2 k_{\ell_I} - \frac{x_1(\sigma k_\ell)}{J_1^i} I\right],\qquad k_\ell = k_{\ell_R} + ik_{\ell_I}\ .$$

Equivalently μ^{ij} satisfies

$$\mu^{ij}(x_0,x,k) = \zeta^{ij} + \frac{\operatorname{sgn}(\sigma_I J_1^i)}{2\pi i}\int_{R_{n+1}} d\xi_0 d\xi\left(\frac{1}{(2\pi)^{n-1}}\int_{R_{n-1}} dm^2 e^{i\alpha^i(x-\xi,m)}\right)$$

$$\times\frac{e^{i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,k)}(q\mu)^{ij}(\xi_0,\xi,k)}{x_1-\xi_1-\sigma J_1^i(x_0-\xi_0)}$$

where

$$dm^2 \doteq dm_2\ldots dm_n,\qquad \alpha^i(x,m) \doteq \sum_{\ell=2}^{n} m_\ell\left(x_\ell - x_1\frac{J^i}{J_1^i}\right)\ .$$

To derive the above note that the exponential of the second term of the right hand side of equation (.) involves

$E^{ij} \doteq i\sum_{\ell=1}^{n} m_\ell(x_\ell-\xi_\ell) - i\sigma\sum_{\ell=1}^{n}[J_\ell^i m_\ell + (J_\ell^i - J_\ell^j)k_\ell](x_0-\xi_0)$. The real

part of E^{ij} is given by $E_R^{ij} = \sum_{\ell=1}^{n}[\sigma_I J_\ell^i m_\ell + (J_\ell^i - J_\ell^j)(\sigma k_\ell)_I](x_0-\xi_0) \mp \sigma_I J_1^i m_1 + \tilde{E}_R^{ij}$.

The second term of the right hand side of equation (.) also involves the integral $\int_{-\infty}^{x_0} d\xi_0 \int_{-\infty} dm_1$, which equals $\int_{-\infty}^{x_0} d\xi_0 \int_{-\infty}^{M_1} dm_1 - \int_{x_0}^{\infty} d\xi_0 \int_M^{\infty} dm_1 +$ $+ \int_{R_1} d\xi_0 \int_M dm_1$, for arbitrary M_1. Since the third term above can be canceled out of (.) with an appropriate choice of $A(m,k)$, it follows that one can always achieve boundness of μ for all complex values of k: Choose M_1 such that E_R^{ij} is less than zero in $\int_{-\infty}^{x_0} d\xi_0$ and greater than zero in $\int_{x_0}^{\infty} d\xi_0$, i.e. $M_1 = -\tilde{E}_R^{ij}/\sigma_I J_1^i$ for $\sigma_1 J_1^i > 0$ (otherwise change sign).

The m_1 integration can be performed explicitly: The coefficient of m_1 in E^{ij} is $i(x_1-\xi_1) - i\sigma J_1^i(x_0 - \xi_0)$, hence this quantity will appear in the denominator. Also E^{ij} evaluated at $m_1 = M_1$ becomes $i \sum_{\ell=2}^{n} m_\ell(x_\ell-\xi_\ell) -$ $- (x_1-\xi_1)J_\ell^i/J_1^i + i\beta^{ij}(x_0-\xi_0, x_1-\xi_1, k)$. Hence (.) yields equation (.) by using the fact that the integral over dm^2 is a product of ζ functions with arguments $\xi_\ell = x_\ell - (x_1-\xi_1)J_\ell^i/J_1^i$.

<u>Remarks</u>.

1. Equation (.) with n=1 is equivalent to the analogous one of 2-spatial dimensions, e.g. equation (.) of [18]. Equation (.) actually appears simpler because the m_1 integration was not carried out in the 2-spatial dimensional case.

2. Equation (.) is also equivalent to that of [28]. The only difference is that the exponential of [28] involves $(x_\ell-\xi_\ell)/J_\ell^i$ instead of $(x_1-\xi_1)/J_1^i$ of (.). However, these two terms are equal due to the existence of the underlying ζ functions.

3. By letting $x_\ell \to x_\ell + J^i_\ell x_1/J^i_1$, $\ell = 2,\dots,n$ in equation (.) one may obtain a more symmetric equation for μ^{ij}:

$$\mu^{ij}(x_0,x_1,x_2 + \frac{J^i_2 x_1}{J^i_1},\dots,x_n + \frac{J^i_n x_1}{J^i_1}) = \zeta^{ij} + \tilde{g}^{ij}(q\mu)^{ij}$$

$$\times(\xi_0,\xi_1,\ x_2 + \frac{J^i_2}{J^i_1}\xi_1,\dots,x_n + \frac{J^i_n}{J^i_1}\xi_1,\ k),$$

where $\tilde{g}^{ij}$ is derived in (.).

4. Equation (.) suggests that $\mu^{ij}(x_0,x,k) = \mu^{ij}(x_0,x_1,x_2 - x_1\frac{J^i_2}{J^i_1},\dots,x_n - \frac{x_1 J^i_n}{J^i_1},k)$. It also suggests that in the proper coordinate system equation (.) should be in some sense reducible to only 2-spatial dimensions. This is indeed the case: Equation (.) in component form becomes

$$\mu^{ij}_{x_0} + \sigma\sum_{\ell=1}^{n} J^i_\ell \mu^{ij}_{x_\ell} + i\sigma\sum_{\ell=1}^{n} k_\ell(J^i_\ell - J^j_\ell)\mu^{ij} = (q\mu)^{ij}.$$

Let

$$\Xi_0 = x_0,\quad \Xi_1 = x_1,\quad \Xi^i_r = x_r - x_1\frac{J^i_r}{J^i_1},\qquad r = 2,\dots,n,$$

i.e. $\partial/\partial x_0 = \partial/\partial\Xi_0$, $\partial/\partial x_r = \partial/\partial\Xi^i_r$, $r = 2,\dots,n$, $\partial/\partial x_1 = \partial/\partial\Xi_1 - \sum_{r=2}^{n} J^i_r\partial/J^i_1\partial\Xi^i_r$. Then (.) yields

$$\mu^{ij}_{\Xi_0} + \sigma J^i_1\mu^{ij}_{\Xi_1} + i\sigma\sum_{\ell=1}^{n} k_\ell(J^i_\ell - J^j_\ell)\mu^{ij} = (q\mu)^{ij}\ .$$

5. Equation (.), as well as equations (.), (.) indicate that the direct problem associated with equation (.) is in some sense 2-spatial dimensional. However, the 2-spatial dimensional results are not

directly applicable due to the shifting in the arguments. Let us illustrate this for the 2 x 2 case in 3-dimensions:

$$\mu^{11}(x_0,x_1,x_2) = 1 + \tilde{g}^{11}(q^1\mu^{21})(\xi_0,\xi_1,\ x_2 - (x_1-\xi_1)\ \frac{J_2^1}{J_1^1})$$

$$\tilde{\mu}^{21}(x_0,x_1,x_2) = \tilde{g}^{21}(q^2\mu^{11})(\xi_0,\xi_1,\ x_2 - (x_1-\xi_1)\ \frac{J_2^2}{J_1^2}).$$

Clearly μ^{11} appears with different arguments in the two equations. However, one may still obtain a solution by iteration. The same is true for the equations corresponding to (.).

Proposition 3.2.

a) The function β^{ij} defined by (.) satisfies

$$\frac{\partial}{\partial\bar{k}_p} e^{i\beta^{ij}(x_0,x_1,k)} = \frac{\bar{\sigma}}{2J_1^i\sigma_I}(x_1 - x_0J_1^i)(J_p^i - J_p^j),$$

and

$$\beta^{\ell j}(k) - \beta^{ij}(k) = \beta^{\ell i}(k^{ij}(k)),\ k_1^{ij} = (k_{1_R} - \sum_{\ell=1}^{n}(\frac{\sigma_R}{\sigma_I}k_{\ell_I} + k_{\ell_R})\frac{J_\ell^i - J_\ell^j}{J_1^i},\ k_{1_I}),$$

$$k_2^{ij} = k_2,\ldots,k_n^{ij} = k_n,$$

where $\beta^{ij}(k)$ denotes $\beta^{ij}(x_0,x_1,k)$ and $k_1^{ij} = k_{1_R}^{ij} + ik_{1_I}^{ij} = (k_{1_R}^{ij}\ ,\ k_{1_I}^{ij})$.

b) The functions β^{ij}, α^i defined by (.), (.) respectively satisfy

$$\alpha^\ell(m) + \beta^{\ell j}(k) - \alpha^i(M) - \beta^{ij}(k) = \alpha^\ell(m-M) + \beta^{\ell i}(\lambda^{ij}(k,M)),$$

where

$$\lambda_1^{ij} = (k_{1_R}^{ij} - \sum_{\ell=2}^{n} M_\ell \frac{J_\ell^i}{J_1^i}, k_{1_I}),\quad \lambda_r^{ij} = (k_{r_R} + M_r,\ k_{r_I}),\quad r = 2,\ldots,n.$$

To derive equation (.) just use $\partial/\partial\bar{k} = \frac{1}{2}\partial/\partial k_R - \frac{1}{2i}\partial/\partial k_I$. To derive equation (.) note that

$$\beta^{\ell j}(k) - \beta^{ij}(k) = \sum_{r=1}^{n} \frac{1}{\sigma_I} [(J_r^\ell - J_r^i) x_0 |\sigma|^2 k_{r_I} - (\frac{J_r^\ell}{J_1^\ell} - \frac{J_r^j}{J_1^\ell} - \frac{J_r^i}{J_1^i} + \frac{J_r^j}{J_1^i}) x_1 (\sigma k_r)_I].$$

But

$$\frac{J_r^\ell}{J_1^\ell} - \frac{J_r^j}{J_1^\ell} - \frac{J_r^i}{J_1^i} + \frac{J_r^j}{J_1^i} = \frac{J_r^\ell - J_r^i}{J_1^\ell} - \frac{(J_1^\ell - J_1^i)}{J_1^\ell} \frac{(J_r^i - J_r^j)}{J_1^i} .$$

Thus

$$\beta^{\ell j}(k) - \beta^{ij}(k) = \sum_{r=1}^{n} \frac{1}{\sigma_I} [(J_r^\ell - J_r^i) x_0 |\sigma|^2 k_{r_I} - \frac{(J_r^\ell - J_r^i)}{J_1^\ell} x_1 (\sigma k_r)_I] +$$

$$\frac{1}{\sigma_I} \frac{(J_1^\ell - J_1^i)}{J_1^\ell} x_1 \sum_{r=1}^{n} \frac{(J_r^i - J_r^j)}{J_1^i} \sigma k_{r_I} .$$

Hence $\beta^{\ell j}(k) - \beta^{ij}(k) = \beta^{\ell i}(k^{ij})$, where all k's are invariant except k_1 which satisfies $k_{1_I}^{ij} = k_{1_I}$, $(\sigma k_1^{ij})_I = (\sigma k_1)_I - \sum_{r=1}^{n} (\sigma k_r)_I (J_r^i - J_r^j)/J_1^i$.

To derive equation (.) note that its left hand side equals

$$\sum_{r=2}^{n} (m_r - M_r)(x_r - x_1 \frac{J_r^\ell}{J_1^\ell}) + \sum_{r=1}^{n} (J_r^\ell - J_r^i) \frac{x_0 |\sigma|^2 k_{r_I}}{\sigma_I}$$

$$- \frac{(J_1^\ell - J_1^i)}{J_1^\ell} x_1 [\sum_{r=2}^{n} - \frac{(J_r^i - J_r^j)}{J_1^i} \frac{(\sigma k_r)_I}{\sigma_I} + \frac{(\sigma k_1)_I}{\sigma_I} - \sum_{r=2}^{n} \frac{J_r^i}{J_1^i} M_r] -$$

$$- \sum_{r=2}^{n} \frac{J_r^\ell - J_r^i}{J_1^\ell} x_1 [\frac{(\sigma k_r)_I}{\sigma_I} + M_r].$$

Hence, equation (.) follows where λ^{ij} is defined by $\lambda_I^{ij} = k_I$ for all k's, $(\sigma \lambda_r^{ij})_I = (\sigma k_r)_I + M_r$, $r = 2,\ldots,n$, $(\sigma \lambda_1^{ij})_I = (\sigma k_1^{ij})_I - \sum_{r=2}^{n} M_r J_r^i / J_1^i$.

Using the above relationships, $\partial\mu/\partial\bar{k}_p$, i.e. the departure from holomorphicity of the eigenfunction μ can be evaluated:

Proposition 3.3.

Let μ^{ij} be defined by equation (.). Then

$$\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k) = \sum_{i,j}\gamma^i(J_p^i-J_p^j)e^{i\beta^{ij}(x_0,x_1,k)}\frac{1}{(2\pi)^{n-1}}$$

$$\times\int_{R_{n-1}} dm^2 e^{i\alpha^i(x,m)}T^{ij}(k,m)\mu(x_0,x,\lambda^{ij}(k,m))E_{ij},$$

where β^{ij}, α^i, λ^{ij} are defined by (.), (.), (.) respectively, E_{ij} is an N x N matrix with zeros in all its entries except the ij=k which equals one, and γ^i, T^{ij} are given by

$$\gamma^i \doteq \frac{\bar{\sigma}}{4\pi i|J_1^i\sigma_I|}, \quad T^{ij}(k,m) \doteq \int_{R_{n+1}} d\xi_0 d\xi e^{-i\beta^{ij}(\xi_0,\xi_1,k)-i\alpha^i(\xi,m)}(q\mu)^{ij}(\xi_0,\xi,k).$$

To derive equation (.) note that $\partial\mu^{ij}/\partial\bar{k}_p$ satisfies the same equation as μ^{ij} where the forcing ζ^{ij} is replaced by $\gamma^i(J_p^i - J_p^j)\exp[i\beta^{ij}(x_0,x_1,k)]\int_{R_{n-1}} dM^2\exp[i\alpha^i(x,M)T^{ij}(k,M)/(2\pi)^{n-1}$. Using $\mu = \sum_{i,j}\mu^{ij}E_{ij}$ it follows that the forcing of the equation satisfied by $\partial\mu/\partial\bar{k}_p$ is given by the above times E_{ij}. Hence

$$\frac{\partial\mu}{\partial\bar{k}_p} = \sum_{i,j}\gamma^i(J_p^i - J_p^j)\frac{1}{(2\pi)^{n-1}}\int_{R_{n-1}} dm^2 T^{ij}(k,m)N_{ij}(x_0,x,k,m),$$

where N_{ij} is a matrix valued function satisfying an equation similar to that of μ but with different forcing:

$$\mu(x_0,x,k) = I + (\tilde{G}\mu)(x_0,x,k), \quad N_{ij} = e^{i(\beta^{ij}+\alpha^i)}E_{ij} + \tilde{G}N_{ij}.$$

Equation (.) implies that $N_{ij} = (\vec{0},\ldots,\vec{N}_{ij}^j,\ldots,\vec{0})$, where the components of the vector $\vec{N}_{ij}$ satisfy

$$N_{ij}^{\ell j}(x_0,x,k,M) = e^{i(\beta^{ij}(x_0,x_1,k) + \alpha^i(x,M))}\xi^{ij} + (\tilde{G}^{\ell j}N_{ij}^{\ell j})(x_0,x,k,M).$$

Multiplying by the negative of the exponential appearing in (.) and using (.) it follows that $N_{ij}^{\ell j}(x_0,x,k,M) = \mu^{\ell i}(x_0,x,\lambda^{ij}(k,MM))$. Hence

$$N_{ij} = (\underbrace{\vec{0},\ldots,\vec{\mu}}_{j}{}^{i}(x_0,x,\lambda^{ij}) = \mu(x_0,x,\lambda^{ij})E_{ij}.$$

Using the above in (.) we obtain (.).

Proposition 3.4.

The potential $q(x_0,x)$ of equation (.) can be reconstructed from

$$q(x_0,x) = \frac{i\sigma}{\pi}\; \hat{J}_p \int_{R_2} dk'_{p_R}\, dk'_{p_I}\, \frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k_1,\ldots,k_{p-1},k'_p,k_{p+1},\ldots,k_n),$$
$$p = 1,\ldots,n,$$

where $\partial\mu/\partial\bar{k}_p$ is evaluated by equation (.) in terms of T^{ij}, μ^{ij}. The eigenfunction μ is reconstructed by

$$\mu(x_0,x,k) = I + \frac{1}{\pi}\int_{R_2} dk'_{p_R} dk'_{p_I}\, \frac{\frac{\partial\mu}{\partial\bar{k}_p}(x_0,x,k_1,\ldots,k'_p,\ldots,k_n)}{k_p - k'_p},\; p=1,\ldots,n.$$

To derive equation (.) inverse $\bar{\partial}\mu$ in the variable k_p. Equation (.) then follows by a similar argument to that used in Proposition 2.3.

Remarks.

1. The forcing of the equation for $\partial\mu/\partial\bar{k}$ can also be written as
$\sum_{i,j} \gamma^i(J_p^i - J_p^j)\exp[i\beta^{ij}(x_0,x_1,k)]t^{ij}(k;x_2-x_1J_2^i/J_1^i,\ldots,x_n-x_1J_n^i/J_1^i)\bar{t}_{ij}$,
where $t^{ij} = \int_{R_{n-1}} dm^2\exp[i\alpha^i(x,m)]T^{ij}(k,m)/(2\pi)^{n-1}$.

2. The results of the Proposition (3.4) can also be directly verified (see below).

B. The Characterization Problem

Equation (.) indicates that there exist n inversion formulae for . Furthermore equation (.) indicates that, unless the inverse data T^{ij} are appropriately constrained, the reconstructed q will depend on k. We now show explicitly that q being independent of k is equivalent to the equality of all the inversion formulae. This is a direct consequence of the following result:

Proposition 3.5.

Let

$$(L_k g)(x_0,x) \doteq \frac{\partial g}{\partial x_0} + \sum_{\ell=1}^{n} J_\ell \frac{\partial g}{\partial x_\ell} + i \sum_{\ell=1}^{n} k_\ell [J_\ell, g],$$

$$(P_{x_0,x,k-k_p} g)(k_p) \doteq \frac{1}{\pi} [\int_{R_2} dk'_{p_R} dk'_{I_R}$$
$$\frac{\times \sum_{i,j} \gamma^i (J_p^i - J_p^j) \frac{1}{(2\pi)^{n-1}} \int_{R_{n-1}} dm^2 e^{i\varepsilon^{ij}(x_0,x,k^{p'},m)} T^{ij}(k^{p'},m) g(x_0,x,\lambda^{ij}(k^{p'},m)) E}{k_p - k'_p}$$

where

$$\varepsilon^{ij}(x_0,x,k,m) \doteq \beta^{ij}(x_0,x_1,k) + \alpha^i(x,m), \quad k^{p'} \text{ denotes } k_1,k_2,\ldots,k'_p,\ldots,k_n.$$

Then

$$[L_k, P_{x_0,x,k-k_p} g](x_0,x,k) = \frac{i\varepsilon}{\pi} \hat{J}_p \int_{R_2} dk'_{p_R} dk'_{I_R} \sum_{i,j} \gamma^i (J_p^i - J_p^j) \frac{1}{(2\pi)^{n-1}} \int_{R_{n-1}}$$
$$dm^2 e^{i\varepsilon^{ij}(x_0,x,k^{p'},m)} T^{ij}(k^{p'},m) g(x_0,x,\lambda^{ij}(k^{p'},m)) E_{ij}.$$

To derive equation (.) note that the term $L_k(P_{x_0,x,k-k_p}, g)$ involves

$$[i(\frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell})\varepsilon^{ij}(k^{p'})]gE_{ij} + (\frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell})gE_{ij} + i\sigma \sum_{\ell=1}^{n} k_\ell[J_\ell, gE_{ij}],$$

while the term $p_{x_0,x,k-k_p}(L_k g)$ involves

$$(\frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell})gE_{ij} + i\sigma \sum_{\ell=1}^{n} \lambda_\ell^{ij}(k^{p'})[J_\ell, g]E_{ij}.$$

Two of the above expressions cancel out, also since $(gE_{ij})^{i'j'}$ is non-zero only if $j=j'$ in which case equals $g^{i'i}$,

$$\{[(\frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell})\varepsilon^{ij}]gE_{ij}\}^{i'j'} = \{[(\frac{\partial}{\partial x_\ell} + \sigma \sum_{\ell=1}^{n} J_\ell \frac{\partial}{\partial x_\ell})\varepsilon^{ij}]g\}^{i'i} =$$

$$[(\frac{\partial}{\partial x_0} + \sigma \sum_{\ell=1}^{n} J_\ell^{i'})\varepsilon^{ij'}]g^{i'i},$$

$$[J_\ell, gE_{ij}]^{i'j'} = (J_\ell^{i'} - J_\ell^{j'})(gE_{ij})^{i'j'} = (J_\ell^{i'} - J_\ell^{j'})g^{i'i},$$

$$\{[J_\ell, g]E_{ij}\}^{i'j'} = [J_\ell, g]^{i'i} = (J_\ell^{i'} - j_\ell^{i})g^{i'i}.$$

Hence, $[L_k, p_{x_0,x,k-k_p}]g$ involves $i\sigma g^{i'i}$ times

$$\sum_{\ell=1}^{n} k_\ell(J_\ell^{i'} - J_\ell^{j'}) + \sum_{\ell=1}^{n} \frac{J_\ell^{i} - J_\ell^{j}}{\sigma_I}\bar{\sigma}\, k_{\ell_I}^{p'} - J_1^{i'} \sum_{\ell=1}^{n} \frac{J_\ell^{i} - J_\ell^{j}}{J_1^{i}}(k_{\ell_R}^{p'} + \frac{\sigma_R}{\sigma_I} k_{\ell_I}^{p'}) -$$

$$J_1^{i'} \sum_{\ell=2}^{n} m_\ell \frac{J_\ell^{i}}{J_1^{i}} + \sum_{\ell=2}^{n} J_\ell^{i'} m_\ell - \sum_{\ell=2}^{n}(J_\ell^{i'} - J_\ell^{i})(ik_{\ell_I}^{p'} + k_{\ell_R}^{p'} + m_\ell) -$$

$$(J_1^{i'} - J_1^{i})[ik_{1_I}^{p'} + k_{1_R}^{p'} - \sum_{\ell=1}^{n}(k_{\ell_R}^{p'} + \frac{\sigma_R}{\sigma_I} k_{\ell_I}^{p'})\frac{J_\ell^{i} - J_\ell^{j}}{J_1^{i}} - \sum_{\ell=2}^{n} \frac{J_\ell^{i}}{J_1^{i}} m_\ell].$$

The above expression equals

$$\sum_{\ell=1}^{n} k_\ell(J_\ell^{i'} - J_\ell^{j'}) - \sum_{\ell=1}^{n} k_\ell^{p'}(J_\ell^{i'} - J_\ell^{j'}) = (k_p - k_p')(J_p^{i'} - J_p^{j'}),$$

which implies equation (.).

Remarks.

1. The above proposition implies that the direct linearizing method is also valid for the general σ case. The relevant result is directly analogous to Proposition 2.8.

2. $q = [L_k, p_{x_0,x,k-k_p}]\mu$, $p=1,\ldots,n$ where the p^{th} expression is independent of k_p. Suppose that q is independent of k_p, k_n, then $[L_k, p_{x_0,x,k-k_p}]\mu = [L_k, p_{x_0,x,k-k_n}]\mu$. Hence $(p_{x_0,x,k-k_p})\mu = (p_{x_0,x,k-k_r})\mu$, i.e. the p^{th} and the r^{th} inversion formulae are equal q is independent of both k_p, k_r.

Proposition 3.6.

a) Assume that $\partial\mu/\partial\bar{k}_p$ is given by equation (.) and that $T^{ij}(k,m)$ is given by (.). Then

$$L^{ij}_{rp}T^{ij}(k,m) + \sum_{\ell=1}^{n} \frac{\gamma^{\ell}}{(2\pi)^{n-1}} \int_{R_{n-1}} dM^2 T^{i\ell}(\lambda^{\ell j}(k,M), m-M)T^{\ell j}(k,M)$$

$$\times [(J^{\ell}_p - J^j_p)(J^i_r - J^{\ell}_r) - (J^{\ell}_r - J^j_r)(J^i_p - J^{\ell}_p)] = 0,$$

where

$$L^{ij}_{rp} \doteq (J^i_p - J^j_p)\frac{\partial}{\partial\bar{k}_r} - (J^i_r - J^j_r)\frac{\partial}{\partial\bar{k}_p}.$$

b) Assume that $\partial\mu/\partial\bar{k}_p$ is given by equation (.) and that $\partial^2\mu/\partial\bar{k}_r\partial\bar{k}_p$ is symmetric with respect to r, p. Then $T^{ij}(k,m)$ solves (.).

To derive equation (.) note that

$$L^{ij}_{rp}T^{ij} = \int_{R_{n+1}} d\xi_0 d\xi_1 e^{-i\varepsilon^{ij}(\xi_0,\xi,k,m)} \{q[(J^i_p - J^j_p)\frac{\partial\mu(k)}{\partial\bar{k}_r} - (J^i_r - J^j_r)\frac{\partial\mu(k)}{\partial\bar{k}_p}\}^{ij}$$

$$= \frac{1}{(2\pi)^{n-1}} \int_{R_{n+1}} d\xi_0 d\xi_1 e^{-i\varepsilon^{ij}(\xi_0,\xi,k,m)} \int_{R_{n-1}} dM^2 \sum_{\ell,j'} \gamma^\ell e^{i\varepsilon^{\ell j'}(\xi_0,\xi,k,M)}$$

$$T^{\ell j'}(k,M)[(J_p^i-J_p^j)(J_r^\ell-J_r^{j'})-(J_r^i-J_r^j)(J_p^\ell-J_p^{j'})]q\mu(\lambda^{\ell j'}(k,M)E_{j'})^{ij}.$$

Since $(\mu E_{\ell j'})^{ij}$ is non-zero only if $j = j'$, evaluate the above at $j = j'$. Also equation (.) implies $-\varepsilon^{ij'}(k,M) + \varepsilon^{\ell j'}(k,M) = -\beta^{i\ell}(\lambda^{\ell j'}(k,M)) - \alpha^\ell(m-M)$, and since $\int_{R_{n+1}} d\xi_0 d\xi_1 \exp[-i\beta^{\ell j'}(\lambda^{\ell j'}(k,M))]\exp[-i\alpha^\ell(m-M)]$ $\times(q\mu)^{\ell i}(\lambda^{\ell j'}(k,M)) = T^{\ell i}(\lambda^{\ell j'}(k,M),\ m-M)$, equation (.) follows.

To derive the second statement of Proposition (3.6) first note

$$L_{rp}^{ij}\, e^{i\beta^{ij}(k)} = 0, \quad L_{rp}^{ij}\, \mu(\lambda^{ij}(k,m)) = L_{rp}^{ij}\, \mu(k),$$

$$\beta^{ij}(k) + \alpha^i(m) + \beta^{i'i}(\lambda^{ij}(k,m) + \alpha^{i'}(M) = \beta^{i'j}(k) + \alpha^{i'}(M+m),$$

$$\times\ \lambda^{i'i}(\lambda^{ij}(k,m),M) = \lambda^{i'j}(k,m+M).$$

Equation (.) follows from (.). Equation (.) means that, with respect to the operator $L_{r,p}$, $\mu(\lambda^{ij})$ should be treated as if its k's were not shifted; it is an obvious consequence of the definition of λ^{ij}. To derive equation (.) use (.) to substitute for $\beta^{i'i}(\lambda^{ij}(k,m)) = \alpha^{i'}(M) + \beta^{i'j}(k) - \alpha^i(m) - \beta^{ij}(k) - \alpha^{i'}(M-m)$. Equation (.) follows from the definition of λ^{ij}:

$$\lambda_{1_R}^{ij}(k,m) = k_{1_R}\frac{J_1^j}{J_1^i} - \frac{\sigma_R}{\sigma_I}k_{1_I}\frac{(J_1^i - J_1^j)}{J_1^i} - \sum_{\ell=2}^n (k_{\ell_R} + \frac{\sigma_R}{\sigma_I}k_{\ell_I})\frac{J_\ell^i-J_\ell^j}{J_1^i} - \sum_{\ell=2}^n m_\ell \frac{J_\ell^i}{J_1^i}$$

Hence

$$\lambda_{1_R}^{i'i}(\lambda^{ij}(k,m),M) = \frac{J_1^i}{J_1^{i'}}\{k_{1_R}\frac{J_1^j}{J_1^i} - \frac{\sigma_R}{\sigma_I}k_{1_I}(\frac{J_1^i-J_1^j}{J_1^i}) -$$

$$\sum_{\ell=2}^n (k_{\ell_R} + \frac{\sigma_R}{\sigma_I}k_{\ell_I})\frac{(J_\ell^i-J_\ell^j)}{J_1^i} - \sum_{\ell=2}^n m_\ell\frac{J_\ell^i}{J_1^i}\}$$

$$- \frac{\sigma_R}{\sigma_I} k_{1_I} \left(\frac{J_1^{i'} - J_1^{i}}{J_1^{i'}}\right) - \sum_{\ell=2}^{n} (k_{\ell_R} + m_\ell + \frac{\sigma_R}{\sigma_I} k_{\ell_I}) \left(\frac{J_\ell^{i'} - J_\ell^{i}}{J_1^{i'}}\right) - \sum_{\ell=2}^{n} M_\ell \frac{J_\ell^{i'}}{J_1^{i'}}$$

$$= k_{1_R} \frac{J_1^{j}}{J_1^{i'}} - \frac{\sigma_R}{\sigma_I} k_{1_I} \left(\frac{J_1^{i'} - J_1^{j}}{J_1^{i'}}\right) - \sum_{\ell=2}^{n} (k_{\ell_R} + \frac{\sigma_R}{\sigma_I} k_{\ell_I}) \left(\frac{J_\ell^{i'} - J_\ell^{j}}{J_1^{i'}}\right) -$$

$$\sum_{\ell=2}^{n} (m_\ell + M_\ell) \frac{J_\ell^{i'}}{J_1^{i'}} = \lambda_{1_R}^{ij} .$$

Let

$$\Delta\mu \doteq \frac{\partial^2 \mu}{\partial \bar{k}_r \partial \bar{k}_p} - \frac{\partial^2 \mu}{\partial \bar{k}_p \partial \bar{k}_r} = \frac{1}{(2\pi)^{n-1}} \sum_{i,j} \int_{R_{n-1}} dm^2 \gamma^i L_{rp}^{ij} \{e^{i\varepsilon^{ij}(k,m)} T^{ij}(k,m) \mu(\lambda^{ij}) E_{ij}\}.$$

Using (.) it follows that

$$\Delta\mu = \frac{1}{(2\pi)^{n-1}} \sum_{i,j} \int_{R_{n-1}} dm^2 \gamma^i e^{i\varepsilon^{ij}(k,m)} (L_{rp}^{ij} T^{ij}) \mu(\lambda^{ij}) E_{ij} +$$

$$\frac{1}{(2\pi)^{2n-2}} \sum_{i,j,i',j'} \int_{R_{2n-2}} dm^2 dM^2 \gamma^i \gamma^{i'} e^{i[\varepsilon^{ij}(k,m) + \varepsilon^{i'j'}(\lambda^{ij}(k,m),M)]} T^{ij}(k,m)$$

$$T^{i'j'}(\lambda^{ij}(k,m),M) + \frac{1}{(2\pi)^{2n-2}} \sum_{i,j,i',j'} \int_{R_{2n-2}} dm^2 dM^2 \gamma^i \gamma^{i'}$$

$$\times e^{i[\varepsilon^{ij}(k,m) + \varepsilon^{i'j'}(\lambda^{ij}(k,m),M)]} T^{ij}(k,m) T^{i'j'}(\lambda^{ij}(k,m),M) \pi \mu(\lambda^{i'j'}(\lambda^{ij}(k,m),M)$$

$$\times E_{i'j'} E_{ij}$$

where $\pi \doteq (J_p^i - J_p^j)(J_r^{i'} - J_r^{j'}) - (J_r^i - J_r^j)(J_p^{i'} - J_p^{j'})$. Since $E_{ij}E_{ij}$ is non zero only if $i=j$ in which case equals $E_{i'j}$ it follows that the above should be investigated at $i=j'$. Then the first term of $\Delta\mu$ involves $\int dp\ \gamma^{i'} \exp[i\varepsilon^{i'j}(k,p)](L_{rp}^{i'j} T^{i'j}(k,p)) \mu(\lambda^{i'j}(k,p))$, while the second term involves (using equations (.) and letting $m+M = p$)

$\iint dp\, dm\, \gamma^{i'}\gamma^{i}\exp[i\epsilon^{i'j}(k,p)]T^{ij}(m)T^{i'i}(\lambda^{ij}(k,m),p-m)\mu(\lambda^{i'j}(k,p)$. Thus $\Delta\mu = 0$ implies the "T equation" (.) (to obtain the identical variables of (.) let $i' \to i$, $i \to \ell$, $p \to m$, $m \to M$).

c. A Special Case and the Hyperbolic Limit

Equations (.), (.) indicate that one may introduce a new parameter $\hat{k}$, which is a combination of k, iff $\sum_1^n (J_\ell^i - J_\ell^j)k_\ell = (J_1^i - J_1^j)\hat{k}$, for all $i,j = 1,\ldots,N$, i.e. iff equation (.) is valid. In this case $\sum_1^n (J_\ell^i - J_\ell^j)k_{\ell_I} = (J_1^i - J_1^j)\hat{k}_I$ and $\sum_1^n (J_\ell^i - J_\ell^j)(\sigma k_\ell)_I/J_1^i = (J_1^i - J_1^j)(\sigma\hat{k})_I/J_1^i$ and hence $\beta^{ij}(x_0,x_1,k)$ becomes $\beta^{ij}(x_0,x,\hat{k})$.

If N=2, or if equations (.) are valid then equations (.) are always solvable for $\hat{k}$. To fix ideas consider the N=3 case. Then equation (.) is solvable iff

$$\frac{\sum_1^n (J_\ell^1 - J_\ell^2)k_\ell}{J_1^1 - J_1^2} = \frac{\sum_1^n (J_\ell^1 - J_\ell^3)k_\ell}{J_1^1 - J_1^3} = \frac{\sum_1^n (J_\ell^2 - J_\ell^3)k_\ell}{J_1^2 - J_1^3} .$$

However, if equation (.) are valid then

$$\frac{J_\ell^1 - J_\ell^2}{J_1^1 - J_1^2} = \frac{J_\ell^1 - J_\ell^3}{J_1^1 - J_1^3} = \frac{J_\ell^2 - J_\ell^3}{J_1^2 - J_1^3} .$$

Multiplying the above by k_ℓ and summing over ℓ we obtain (.). The general N case is a trivial extension of the above where one uses $(J_\ell^i - J_\ell^j)/(J_1^i - J_1^j) = (J_\ell^i - J_\ell^{j'})/(J_1^i - J_1^{j'})$. From the above it follows that:

Proposition 3.7.

One may always introduce a new $\hat{k} = \sum_1^n (J_\ell^i - J_\ell^j)/J_1^i - J_1^j$ in equation (.), provided that N=2 or the J_ℓ's are constrained according to equation (.). In this case the inverse data depends only on n+1 parameters

$(\hat{k}_R, \hat{k}_I, m_2, \ldots, m_n)$ and the characterization problem is bypassed. This is consistent with the fact that the T equation now

The analytic eigenfunctions $\mu^\pm$ used for the solution of the hyperbolic problem can be obtained as a limiting case of the general σ case: Let $\sigma = -1 + i\sigma_I$, $k = (k_R, \sigma_I k_I)$, $\sigma_I \to 0^+$. Then the limits $k_{1_I} \to \pm\infty$ yields eigenfunctions $\mu^\pm$ respectively, analytic in the variable k_{1_R}. The details can be found in [28].

APPENDIX A

In this appendix we derive equations (.), (.). Equations (.) can be written as

$$\mu^\pm = I + \int_{-\infty}^{x} d\xi_0 \int_{R_{2n}} d\xi dm\, E(x_0-\xi_0, x-\xi, m) e^{i(x_0-\xi_0)k\hat{J}} (q\mu^\pm)(\xi_0,\xi,k) +$$

$$+ \int_{R_1} d\xi_0 \int_{R_{2n}} d\xi_0 dm\, E\, e^{i(x_0-\xi_0)k\hat{J}} \pi_\mp (q\mu^\pm),$$

where $E \doteq \exp[im(x-\xi) + i(x_0 - \xi_0)mJ/(2\pi)^n$. Thus if $\Delta = \mu^+ - \mu^-$ then

$$\Delta = \int_{R_{2n+1}} d\xi_0 d\xi dm\, E\, e^{i(x_0-\xi_0)k\hat{J}} (\pi_+ q\mu^- - \pi_- q\mu^+) +$$

$$+ \int_{-\infty}^{x} d\xi_0 \int_{R_{2n+1}} d\xi dm\, E\, e^{i(x_0-\xi_0)k\hat{J}} (q\Delta)(\xi_0,\xi,k).$$

We wish to prove that Δ equals the right hand side of equation (.), or, substituting this Δ in the above and canceling the exp[-ikx], we need to prove that

$$\int_{R_n} d\lambda \mu^-(x_0,x,\lambda) e^{ix_0\lambda J} f(\lambda,k) e^{i(\lambda-k)x} = \int_{R_{2n+1}} d\xi_0 d\xi dm\, E\, e^{i(x_0-\xi_0)kJ}$$

$$\times (\pi_+ q\mu^- - \pi_- q\mu^+) e^{i\xi_0 kJ} +$$

$$+\int_{-\infty}^{x_0} d\xi_0 \int_{R_{2n}} d\xi E\, e^{i(x_0-\xi_0)kJ} q \int_{R_n} d\lambda \mu^-(\xi_0,\xi,\lambda) e^{i\xi_0\lambda J} f(\lambda,k) e^{i(\lambda-k)\xi}.$$

However, equation $(A.1)^-$ implies

$$\int_{R_n} d\lambda \mu^-(x_0,x,\lambda) e^{ix_0\lambda J} f(\lambda,k) e^{i(\lambda-k)x} = \int_{R_n} d\lambda e^{ix_0\lambda J} f(\lambda,k) e^{i(\lambda-k)x}$$

$$+\int_{-\infty}^{x_0} d\xi_0 \int_{R_{3n}} d\lambda d\xi dm\, E\, e^{i(x_0-\xi_0)\lambda J} q\mu^- e^{-i(x_0-\xi_0)\lambda J + ix_0\lambda J} f(\lambda,k) e^{i(\lambda-k)x} .$$

The integrals involving $\int_{-\infty}^{x_0}$ of equations (A.2), (A.3) are equal. To prove this let $m+k \to \bar{m}$, $m+\lambda \to \bar{m}$, alternatively use the following property of E:

$$Ee^{i(x_0-\xi_0)kJ} Ae^{i(\lambda-k)\xi} = Ee^{i(x_0-\xi_0)kJ} e^{i(\lambda-k)} A^{(x+J(x_0-\xi_0))} = Ee^{i(x_0-\xi_0)\lambda J} Ae^{i(\lambda-k)x} .$$

Hence (A.2), (A.3) imply (by letting $\lambda \to \bar{m}$ in the second integral of (A.3)),

$$\int_{R_n} d\bar{m} e^{ix_0\bar{m}J} f(\bar{m},k) e^{ix(\bar{m}-k)} -$$

$$(2\pi)^{-1/n} \int_{R_{3n+1}} d\lambda d\xi_0 d\, dm e^{i(x_0-\xi_0)(\lambda+m)J + i(x-\xi)m} \pi_+(q\mu^-)(\xi_0,\xi,\lambda) e^{i\xi_0\lambda J}$$

$$\times f(\lambda,k) e^{i(\lambda-k)x} = \frac{1}{(2\pi)^n} \int_{R_{2n+1}} d\lambda d\xi_0 dm\, e^{i(x_0-\xi_0)(k+m)J + i(x-\xi)m}$$

$$\times (\pi_+ q\mu^- - \pi_- q\mu^+)(\xi_0,\xi,k) e^{i\xi_0 kJ} .$$

Multiplying and dividing the second and third terms of the above equation by $\exp[i\lambda(x-\xi)]$ and $\exp[ik(x-\xi)]$ respectively, we obtain

$$f(\bar{m},k) = \int_{R_n} d\lambda T_+(\bar{m},\lambda) f(\lambda,k) = T_+(\bar{m},k) - T_-(\bar{m},k).$$

APPENDIX B

We now verify that the second and third terms of (.) are equal, i.e.

$$\int_{R_n} d\lambda\{\mu_L(x,\lambda)E(\lambda,x)T_+(\lambda,k)E(-k,x) + \mu^-(x,\lambda)E(\lambda,x)\Lambda(\lambda,k)E(-k,x)\} = 0,$$

where $E(k,x) = \exp[ik(x+x_0J)]$. Substituting for μ^- in the above by (.) and replacing Λ by (.) we obtain

$$\int_{R_n} d\lambda\{\mu_L(x,\lambda)E(\lambda,x)\int_{R_n} dmT_+(\lambda,m)\Lambda(m,k) - \int_{R_n} dr\mu_L(x,r)E(r,x)T_+(r,\lambda)\Lambda(\lambda,k)\}E(-k,x)=0,$$

which is obviously true by letting $r \to m$.

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A Multidimensional Inverse-Scattering Method

By *Adrian I. Nachman** and *Mark J. Ablowitz*

A formal solution of the inverse scattering problem for the n-dimensional time-dependent and time-independent Schrödinger equations is given. Equations are found for reconstructing the potential from scattering data purely by quadratures. The solution also helps elucidate the problem of characterizing admissible scattering data.

In this note we present an inverse-scattering formalism which applies to a variety of multidimensional problems. Our procedure is based upon the use of the so-called "$\bar{\partial}$ method," first introduced in the study of inverse scattering problems on the line by Beals and Coifman [1, 2] and successfully extended to two-dimensional problems in [3]. This method gives a systematic procedure for finding not only linear integral equations to reconstruct the eigenfunctions and the potential, but also necessary conditions which the scattering data must satisfy. These characterization conditions also turn out to provide an alternative way to reconstruct the potential directly from the scattering data purely by quadratures. Moreover, these conditions may help explain why there are so few nonlinear evolution equations in dimensions higher than $2+1$ known to be solvable by the inverse scattering transform. We give here our results for the time-dependent and time-independent Schrödinger operators (for earlier treatments of the time-dependent Schrödinger operator in one dimension see [4] and [5]); similar results for first-order systems will appear separately.

Our approach is to first study the operator $L_\sigma = \sigma\,\partial/\partial t + \Delta - v(t,x)$ with $\sigma = \sigma_R + i\sigma_I$ any complex number with $\sigma_R \neq 0$ [here $(t,x) \in \mathbb{R}\times\mathbb{R}^n$ and $\Delta =$

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$\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2]$. The case $\sigma = i$ will then be arrived at by a limiting procedure. One of the advantages of this route is that it always leads us to Green's functions with the suitable symmetry properties. For the classical time-independent Schrödinger equation our procedure yields precisely the Green's function introduced by Faddeev [6]. Subsequently Faddeev [7] and independently Newton [8] used this Green's function in their study of the three-dimensional inverse problem. Although we arrive at the reconstruction by a different route, our characterization conditions parallel those of [7]. More recently, in a series of important papers [9], Newton has carefully studied this problem with new methods having the classical Green's function as their starting point. We will indicate how our characterization equations compare with his "miraculous" condition.

To begin, we assume $\sigma_R \neq 0$ and look for eigenfunctions of L_σ of the form $\mu \exp(ik \cdot x + k^2 t/\sigma)$ where $k \in \mathbb{C}^n$, $k \cdot x = k_R \cdot x + ik_I \cdot x$, $k^2 = k_1^2 + \cdots + k_n^2$, and find that μ satisfies

$$\sigma\mu_t + \Delta\mu + 2ik \cdot \nabla\mu - v\mu = 0. \tag{1}$$

The Green's function we use is given by

$$G_\sigma(t,x,k) = \frac{1}{(2\pi)^{n+1}} \iint \frac{e^{i(t\tau + x\cdot\xi)}}{i\sigma\tau - \xi^2 - 2k\cdot\xi}\, d\xi\, d\tau. \tag{2}$$

A solution μ_σ of (1) is obtained by solving the integral equation $\mu_\sigma = 1 + \tilde{G}_\sigma(v\mu_\sigma)$ with $\tilde{G}_\sigma$ the integral operator whose kernel is $G_\sigma(t - t', x - x', k)$. In this mostly formal presentation we assume for simplicity that $v(t, x)$ is such that the integral equation defining μ_σ has a solution for every $k \in \mathbb{C}^n$.

Differentiating the integral equation for μ_σ with respect to $\bar{k}_j$ produces another solution of (1), which can be expressed nonlocally in terms of μ using the important symmetry property of G:

$$\exp(-i\beta_\sigma(t, x, k_R, \xi))G_\sigma(t, x, k_R, k_I) = G_\sigma(t, x, \xi, k_I)$$
$$\text{whenever } \left(\xi + \frac{\sigma_I}{\sigma_R}k_I\right)^2 = \left(k_R + \frac{\sigma_I}{\sigma_R}k_I\right)^2 \tag{3}$$

with $\beta_\sigma(t, x, k_R, k_I, \xi) = (x + 2\frac{t}{\sigma_R}k_I)\cdot(\xi - k_R)$. We obtain

$$\frac{\partial\mu_\sigma}{\partial\bar{k}_j}(t, x, k)$$
$$= \frac{-1}{(2\pi)^n|\sigma_R|}\int e^{i\beta_\sigma(t,x,k,\xi)}(\xi_j - k_{R_j})\,\delta\left(\left(\xi + \frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R + \frac{\sigma_I}{\sigma_R}k_I\right)^2\right)$$
$$\times T_\sigma(k_R, k_I, \xi)\mu_\sigma(t, x, \xi, k_I)\, d\xi. \tag{4}$$

where the scattering data are found to be

$$T_\sigma(k_R, k_I, \xi) = \int\int \exp[-i\beta_\sigma(t, x, k, \xi)]\, v(t, x)\mu_\sigma(t, x, k)\, dt\, dx.$$

The general Cauchy integral formula applied to the variable k_j together with the fact that $\mu_\sigma \to 1$ as $|k_j| \to \infty$ allows us to write (4) in integral form as

$$\mu_\sigma(t, x, k_R, k_I) = 1 - \frac{1}{\pi(2\pi)^n|\sigma_R|}\int\int\int \frac{(\xi_j - k'_{R_j})e^{i\beta_\sigma(t, x, k', \xi)}}{k_{R_j} - k'_{R_j} + i(k_{I_j} - k'_{I_j})}$$
$$\times \delta\left(\left(\xi + \frac{\sigma_I}{\sigma_R}k'_I\right)^2 - \left(k'_R + \frac{\sigma_I}{\sigma_R}k'_I\right)^2\right)$$
$$\times T_\sigma(k'_R, k'_I, \xi)\mu_\sigma(t, x, \xi, k'_I)\, dk'_{R_j}\, dk'_{I_j}\, d\xi. \qquad (5)$$

where in the integral $k'_R = (k_{R_1}, \dots, k'_{R_j}, \dots, k_{R_n})$, and similarly for k'_I. Equation (5) is the reconstruction equation for μ. Comparing the large-k_j behavior in (5) and (1) yields

$$v(t, x) = \frac{2i}{\pi}\frac{\partial}{\partial x_j}\int\int \frac{\partial\mu}{\partial \bar{k}_j}(t, x, k_R, k_I)\, dk_{R_j}\, dk_{I_j}. \qquad (6)$$

The fact that (when $n > 1$) the right side of (6) does not depend on k_i ($i \neq j$) or j is analogous to the miracle in the procedure of Newton; however in our case it may be deduced from the characterization equations which follow.

Our "characterization equations" can be found from the compatibility conditions $\partial^2\mu/\partial\bar{k}_i\partial\bar{k}_j = \partial^2\mu/\partial\bar{k}_j\partial\bar{k}_i$; differentiating (4) and integrating by parts, we obtain equations which suggest that T_σ satisfies

$$(\xi_j - k_{R_j})\left(\frac{\partial T_\sigma}{\partial \bar{k}_i} + \frac{1}{2}\frac{\partial T_\sigma}{\partial \xi_i}\right) - (\xi_i - k_{R_i})\left(\frac{\partial T_\sigma}{\partial \bar{k}_j} + \frac{1}{2}\frac{\partial T_\sigma}{\partial \xi_j}\right) = N_{ij}[T_\sigma], \quad (6\mathrm{a}_{ij})$$

$$N_{ij}[T_\sigma](k, \xi) = \frac{1}{(2\pi)^n|\sigma_R|}\int\left[(\xi'_j - k_{R_j})(\xi_i - \xi'_i) - (\xi'_i - k_{R_i})(\xi_j - \xi'_j)\right]$$
$$\times \delta\left(\left(\xi' + \frac{\sigma_I}{\sigma_R}k_I\right)^2 - \left(k_R + \frac{\sigma_I}{\sigma_R}k_I\right)^2\right)$$
$$\times T_\sigma(k_R, k_I, \xi')T_\sigma(\xi', k_I, \xi)\, d\xi'. \qquad (6\mathrm{b}_{ij})$$

It is also straightforward to check (6) directly from the definition of T_σ. To avoid redundancies we keep only the equations (6_{1j}). We now parametrize the surface

$$\left(\xi + \frac{\sigma_I}{\sigma_R} k_I\right)^2 = \left(k_R + \frac{\sigma_I}{\sigma_R} k_I\right)^2$$

in (k, ξ) space in terms of $(\chi, w_0, w) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$ as follows: $k_{R_1} = \sum_{j=2}^n w_j \chi_{R_j} - w_1/2 - \sigma_I w_0 w_1/(2w^2)$, $k_{R_j} = -w_1 \chi_{R_j} - w_j/2 - \sigma_I w_0 w_j/(2w^2)$, $j \geqslant 2$; $k_{I_1} = \sum_{j=2}^n w_j \chi_{I_j} + \sigma_R w_0 w_1/(2w^2)$, $k_{I_j} = -w_1 \chi_{I_j} + \sigma_R w_0 w_j/(2w^2)$; $\xi_1 = \sum_{j=2}^n w_j \chi_{R_j} + w_1/2 - \sigma_I w_0 w_1/(2w^2)$, $\xi_j = -w_1 \chi_{R_j} + w_j/2 - \sigma_I w_0 w_j/(2w^2)$. In the new variables, (6) can be written in integral form as

$$I_j[T_\sigma](\chi, w_0, w) \equiv T_\sigma(\chi, w_0, w) - \frac{1}{\pi} \int\int \frac{N_{1j}[T_\sigma](\chi', w_0, w)}{\chi_{R_j} - \chi'_{R_j} + i(\chi_{I_j} - \chi'_{I_j})} d\chi'_{R_j} d\chi'_{I_j}$$

$$= \hat{v}(w_0, w), \tag{7}$$

where $\chi' = (\chi_2, \dots, \chi'_j, \dots, \chi_n)$ and $\hat{v}(w_0, w) = \int\int e^{-i(tw_0 - x \cdot w)} v(t, x)\, dt\, dx$. We have used the fact that when $w_0 = 2k_I \cdot (\xi - k_R)/\sigma_R$ and $w = \xi - k_R$ are kept fixed, $T(\chi, w_0, w) \to \hat{v}(w_0, w)$ for large χ_j (if $w_1 \neq 0$); this is the analogue of the Born approximation.

It seems reasonable to conjecture that (at least for small perturbations v) the main condition needed to characterize scattering data is that $I_j[T_\sigma](\chi, w_0, w)$ is independent of χ and j and has suitable decay properties in (w_0, w). Moreover, it seems reasonable to solve the (re)construction problem directly from (7): namely, if T_σ is admissible, compute v by taking the inverse Fourier transform of $I(T)$. As opposed to the Born approximation, this formula for reconstruction does not rely exclusively on high-energy values of the scattering data.

Next we treat the case $\sigma = i$ as the limit of the above. The limit of $\mu_\sigma(t, x, k_R, k_I)$ does not appear to provide enough information (when $k_I = 0$) for reconstruction purposes; we consider instead $\mu_\sigma(t, x, k_R, \sigma_R k_I)$. Since

$$G_\sigma(t, x, k_R, \sigma_R k_I) \to G_L(t, x, k_R, k_I)$$

$$= \frac{i \operatorname{sgn}(-t)}{(2\pi)^n} \int e^{-it(\xi^2 + 2k_R \cdot \xi) + ix \cdot \xi} \theta(t(\xi^2 + 2(k_R - k_I) \cdot \xi))\, d\xi$$

as $\sigma \to i$ $(\sigma_R < 0)$, our limiting eigenfunction $\mu_L(t, x, k_R, k_I)$ solves the integral equation $\mu_L = 1 + \tilde{G}_L(v\mu_L)$; $\mu_L(t, x, k_R, k_I)$ is a solution of $i\mu_t + \Delta\mu + 2ik_R \cdot \nabla\mu - v = 0$ for every value of the parameter k_I. Taking limits in (5) yields the

integral equations for reconstructing μ_L:

$$\mu_L(t,x,k_R,k_I) = 1 - \frac{1}{\pi(2\pi)^n}\iiint\left[\frac{\theta(k_I - k'_{I_j})}{k_{R_j} - k'_{R_j} - i0} - \frac{\theta(k'_I - k_I)}{k_{R_j} - k'_{R_j} + i0}\right]$$

$$\times\delta\left((\xi + k'_I)^2 - (k_R + k'_I)^2\right)\mu_L(t,x,\xi,k'_I)$$

$$\times(\xi_j - k'_{R_j})T_L(k'_R,k'_I,\xi)$$

$$\times\exp\left(+i\beta_L(t,x,k'_R,k'_I,\xi)\right)dk'_{R_j}\,dk'_{I_j}\,d\xi, \tag{8}$$

where $\theta(\cdot)$ is the usual Heaviside function, $\beta_L(t,x,k_R,k_I,\xi) = (x + 2tk_I)\cdot(\xi - k_R)$, and the scattering data are given by $T_L(k_R,k_I,\xi) = \iint\exp[-i\beta_L(t,x,k_R,k_I,\xi)]\,v(t,x)\mu_L(t,x,k_R,k_I)\,dt\,dx$. The characterization equations (7) now become

$$I_j[T_L](\chi,w_0,w) = T_L(\chi,w_0,w) - \frac{1}{\pi}\iint\left[\frac{\theta(\chi_{I_j} - \chi'_{I_j})}{\chi_{R_j} - \chi'_{R_j} - i0} - \frac{\theta(\chi'_{I_j} - \chi_{I_j})}{\chi_{R_j} - \chi'_{R_j} + i0}\right]$$

$$\times N_{1,j}[T_L](\chi',w_0,w)\,d\chi'_{R_j}\,d\chi'_{I_j}$$

$$= \hat{v}(w_0,w) \tag{9}$$

with the obvious modifications in the definition of $N_{1,j}[T_L]$ and in the change of variables.

It should be remarked that the above limiting results can also be obtained directly starting from the Green's function G_L (the derivative with respect to the parameter k_I essentially plays the role of $\bar{\partial}$).

When v does not depend on t, the equations above formally yield the following results for the time-independent Schrödinger operator ($n > 1$). Again, all of these results can be independently established without recourse to this derivation (assuming, as before, that the potential is such that there are no exceptional points in our integral equations). The Green's function is now

$$G_s(x,k_R,k_I) = \int_{-\infty}^{\infty} G_L(t,x,k_R,k_I)\,dt$$

$$= -\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\left[\frac{\theta(k_I\cdot\xi)}{\xi^2 - 2k_R\cdot\xi - i0} - \frac{\theta(-k_I\cdot\xi)}{\xi^2 - 2k_R\cdot\xi - i0}\right]d\xi,$$

and the reconstruction equations are

$$\mu_s(x,k_R,k_I) = 1 - \frac{1}{(2\pi)^n}\iiint\left[\frac{\theta(k_{I_i}-k'_{I_i})}{k_{R_i}-k'_{R_i}-i0}+\frac{\theta(k'_{I_i}-k_{I_i})}{k_{R_i}-k'_{R_i}-i0}\right]$$

$$\times\,\delta(k'_I\cdot(\xi-k'_R))\delta(\xi^2-k'^2_R)\mu_s(x,\xi,k'_I)$$

$$\times\,(\xi_j-k'_{R_j})T_s(k'_R,k'_I,\xi)\exp[ix\cdot(\xi-k_R)]\,dk'_{R_i}\,dk'_{I_i}\,d\xi$$

$$(k_R\neq 0,\quad k_I\neq 0),\qquad (10)$$

where $T_s(k_R,k_I,\xi)=\int\exp[-ix\cdot(\xi-k_R)]v(x)\mu_s(x,k_R,k_I)\,dx$. Using the same change of variables as before with $w_0\equiv 0$, we can write the characterization equations as

$$I_j[T_s](\chi,w) = T_s(\chi,w) - \iint\left[\frac{\theta(\chi_{I_i}-\chi'_{I_i})}{\chi_{R_i}-\chi'_{R_i}-i0}+\frac{\theta(\chi'_{I_i}-\chi_{I_i})}{\chi_{R_i}-\chi'_{R_i}-i0}\right]$$

$$\times\,N_{i,j}[T_s](\chi',w)\,d\chi'_{R_i}\,d\chi'_{I_i} = \hat{v}(w), \qquad (11a)$$

where

$$N_{i,j}[T_s](k,\xi) = \frac{1}{(2\pi)^n}\int\left[(\xi'_j-k_{R_j})(\xi_i-\xi'_i)-(\xi'_i-k_{R_i})(\xi_j-\xi'_j)\right]\delta(\xi'^2-k_R^2)$$

$$\times\,\delta(k_I\cdot(\xi'-k_R))T_s(k_R,k_I,\xi')T_s(\xi',k_I,\xi)\,d\xi'. \qquad (11b)$$

If the scattering data T_s are given, then we use (11) to check admissibility and reconstruct v. Suppose, on the other hand, that we are given the scattering amplitude $A(k_R,\xi)=\int\exp[-ix\cdot(\xi-k_R)]\,v(x)\mu_+(x,k_R)\,dx$, with μ_+ corresponding to the classical Green's function

$$G_+(x,k_R) = -\frac{1}{(2\pi)^n}\int\frac{\exp(ix\cdot\xi)\,d\xi}{\xi^2+2k_R\cdot\xi-i0};$$

A is related to T_s via

$$T_s(k_R,k_I,\xi) = A(k_R,\xi) + \frac{i}{(2\pi)^{n-1}}\int\theta(k_I\cdot(k_R-\xi'))\delta(\xi'^2-k_R^2)$$

$$\times\,T_s(k_R,k_I,\xi')A(\xi',\xi)\,d\xi'. \qquad (12)$$

Solving (12) for T_s and checking that $I_j[T_s](\chi, w)$ is independent of χ and j should now be compared to Newton's procedure, where, given A, an integral equation is solved to find a candidate for the potential, which has to turn out independent of certain additional variables.

Finally, note that (11) is equivalent to the following statements together: (i) $\partial T_s / \partial \chi_I = 2\pi i N_{1j}[T_s]$, and (ii) $\lim_{\chi_I \to \pm\infty}(T_s - \hat{v})$ is, as a function of χ_R, the boundary value of an analytic function in the lower (respectively upper) half plane with appropriate boundedness properties. Any solution T_s of (12) can be shown to satisfy (i), while (ii) corresponds to Faddeev's condition for admissibility.

A more detailed study of the equations presented here will be published elsewhere.

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Multidimensional Inverse Scattering for First-Order Systems

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A method for solving the inverse problem for a class of multidimensional first-order systems is given. The analysis yields equations which the scattering data must satisfy; these equations are natural candidates for characterizing admissible scattering data. The results are used to solve the multidimensional N-wave resonant interaction equations.

1. Introduction

The inverse scattering problems for the hyperbolic and elliptic generalizations in the plane of the $m \times m$ AKNS system have been successfully studied in [1] and applied to the linearization of several physically significant nonlinear evolution equations (N-wave interaction, Davey-Stewartson, etc.) in two spatial and one temporal dimensions. We indicate here how the method used in our investigation of the n-dimensional Schrödinger equation [2] can be applied to the study of the inverse problem for the operator in $\mathbb{R}^{n+1}$:

$$L_\sigma = \frac{\partial}{\partial x_0} - \sigma \sum_{l=1}^{n} J_l \frac{\partial}{\partial x_l} - Q(x_0, x). \tag{1}$$

Here J_l are constant real diagonal $m \times m$ matrices (we denote the diagonal entries of J_l by $J_l^1, \ldots, J_l^m$ and assume $J_l^i \neq J_l^j \neq 0$ whenever $i \neq j$); the matrix-

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valued off-diagonal potential $Q = (Q^{ij})$ may depend on x_0 as well as $x = (x_1, \ldots, x_n)$ and $\sigma = \sigma_R - i\sigma_I$ is a complex parameter.

The operator (1) is associated with the nonlinear system

$$\frac{\partial Q^{ij}}{\partial t} = \frac{1}{\sigma} a_{ij} \frac{\partial Q^{ij}}{\partial x_0} - \sum_l \left(a_{ij} J_l^i - B_l^i \right) \frac{\partial Q^{ij}}{\partial x_l} + \frac{1}{\sigma} \sum_l \left(a_{il} - a_{lj} \right) Q^{il} Q^{lj} \tag{2}$$

with

$$a_{ij} = \frac{B_l^j - B_l^i}{J_l^j - J_l^i}, \quad 1 \leqslant l \leqslant n. \qquad \text{for some real } B_l^i,\ 1 \leqslant l \leqslant n,\ 1 \leqslant i \leqslant m. \tag{3}$$

Even though no traditional scattering operator exists in the case $\sigma_I \neq 0$, the so-called $\bar{\partial}$ method (see [2] and references given there) gives a satisfactory definition of scattering data for L_σ, along with a systematic inversion procedure, which we use to solve (2).

A solution of the inverse scattering problem for the hyperbolic case $\sigma_I = 0$ is then obtained by a limiting argument; thus we can avoid a separate study of a Riemann-Hilbert boundary-value problem (which in higher dimensions may also involve some geometric complications). The main advantage of this approach is that it yields (from the compatibility conditions associated with $\bar{\partial}$ in several variables) equations which must be satisfied by the scattering data. In addition to their importance for the problem of characterizing admissible scattering data, these equations have several consequences: (i) they provide a formula for reconstructing the potential from the scattering transform purely by quadratures [in the generic case when no three of the vectors $J^i = (J_1^i, J_2^i, \ldots, J_n^i)$, $1 \leqslant i \leqslant m$, are collinear]; (ii) they show how one can recover the scattering transform from (at least small) data given on certain $(n-1)$-dimensional surfaces ($n-1$ being the number of variables in Q); (iii) they may indicate what other (possibly nonlocal) evolution equations could be solvable with the IST developed here; (iv) they constitute in themselves a new class of multidimensional nonlinear systems of integrodifferential equations which are linearizable.

2. The case $\sigma_I \neq 0$.

We will denote by $k = (k_1, \ldots, k_n) = k_R + ik_I$ a point in $\mathbb{C}^n$ and will often write $f(k)$ instead of $f(k_R, k_I)$ for an arbitrary function of k_R and k_I.

As a first step in the $\bar{\partial}$ procedure we construct a family of solutions of $L_\sigma\psi = 0$ of the form $\psi = \mu(x_0, x, k)\exp[i\sum_{l=1}^n k_l(x_l - \sigma x_0 J_l)]$ with μ bounded; μ will then satisfy the equation

$$\frac{\partial \mu}{\partial x_0} + \sigma \sum_{l=1}^n J_l \frac{\partial \mu}{\partial x_l} - i\sigma \sum_{l=1}^n k_l [J_l, \mu] = Q\mu. \tag{4}$$

The generalized eigenfunctions $\mu_\sigma = (\mu_\sigma^{ij})$ we will work with are obtained by

solving the integral equation $\mu_\sigma = I - \bar{G}_\sigma(Q\mu_\sigma)$, i.e.

$$\mu_\sigma^{ij} = \delta_{ij} + \iint_{\mathbf{R}^{n+1}} G_\sigma^{ij}(x_0 - y_0, x - y, k)(Q(y_0, y)\mu_\sigma(y_0, y, k))^{ij}\, dy_0\, dy. \quad (5)$$

where the Green's function is given by

$$G_\sigma^{ij}(x_0, x, k) = \frac{-i}{(2\pi)^{n+1}} \iint_{\mathbf{R}^{n+1}} \frac{e^{i(x_0\xi_0 + x\cdot\xi)}}{\xi_0 + \sigma\sum_{l=1}^n [J_l^i\xi_l + k_l(J_l^i - J_l^j)]}\, d\xi_0\, d\xi. \quad (6)$$

For brevity we will assume here that Q is such that this integral equation has a bounded solution μ_σ for all $k \in \mathbf{C}^n$.

G_σ can be computed explicitly by contour integration:

$$G_\sigma^{ij}(x_0, x, k) = \frac{\operatorname{sign}(\sigma_I J_1^i)}{2\pi i(x_1 - \sigma J_1^i x_0)} e^{i\alpha_\sigma^{ij}(x_0, x, k)} \prod_{l=2}^n \delta\left(x_l - \frac{J_l^i}{J_1^i}x_1\right) \quad (7)$$

with

$$\alpha_\sigma^{ij}(x_0, x, k) = \sum_{l=1}^n \frac{J_l^i - J_l^j}{\sigma_I}\left(|\sigma|^2 x_0 k_{I_l} - \frac{x_l}{J_l^i}(\sigma k_l)_I\right). \quad (8)$$

The next step is to express $\bar{\partial}\mu$ in terms of μ. We start by writing $\dfrac{\partial G}{\partial \bar{k}_p}$ and hence $\dfrac{\partial \bar{G}}{\partial \bar{k}_p}(Q\mu)$ as a superposition of exponentials:

$$\left(\frac{\partial \bar{G}_\sigma}{\partial \bar{k}_p}(Q\mu_\sigma)\right)^{ij} = \frac{\bar{\sigma}(J_p^i - J_p^j)}{2i|\sigma_I|(2\pi)^n}\int_{\mathbf{R}^n} \delta\left(\sum_{l=1}^n J_l^i\lambda_l\right) e^{i\beta_\sigma^{ij}(x_0, x, k, \lambda)} T_\sigma^{ij}(k, \lambda)\, d\lambda \quad (9)$$

with

$$\beta_\sigma^{ij}(x_0, x, k, \lambda) = \alpha_\sigma^{ij}(x_0, x, k) + \sum_{l=1}^n (x_l - \sigma_R J_l^i x_0)\lambda_l \quad (10)$$

and

$$T_\sigma^{ij}(k, \lambda) = \iint_{\mathbf{R}^{n+1}} e^{-i\beta_\sigma^{ij}(y_0, y, k, \lambda)}(Q(y_0, y)\mu_\sigma(y_0, y, k))^{ij}\, dy_0\, dy. \quad (11)$$

The calculation of $\bar{\partial}\mu$ is then based on the following crucial symmetry property of our Green's function:

$$e^{-i\beta_\sigma^{ij}(x_0, x, k, \lambda)} G_\sigma^{ii}(x_0, x, k) = G_\sigma^{ii}(x_0, x, \hat{k}_\sigma^{ij}(k, \lambda)) \qquad \text{whenever} \quad \sum J_l^i\lambda_l = 0: \quad (12)$$

here $\hat{k}_\sigma^{ij}(k,\lambda)$ is the point in $\mathbb{C}^n$ whose lth component is

$$\left(\hat{k}_\sigma^{ij}(k,\lambda)\right)_l = \frac{J_l^j - J_l^i}{\sigma_I J_l^i}(\sigma k_l)_I + k_l + \lambda_l. \tag{13}$$

Once (12) has been established, it can be shown [assuming that (5) admits no homogeneous solutions] that

$$\frac{\partial \mu_\sigma}{\partial \bar{k}_p}(x_0,x,k) = \sum_{i,j} \frac{\bar{\sigma}\left(J_p^i - J_p^j\right)}{2i|\sigma_I|(2\pi)^n} \int_{\mathbf{R}^n} \delta\left(\sum J_l^i \lambda_l\right) T_\sigma^{ij}(k,\lambda)\, e^{i\beta_\sigma^{ij}(x_0,x,k,\lambda)}$$

$$\times \mu_\sigma\left(x_0,x,\hat{k}_\sigma^{ij}(k,\lambda)\right) E_{ij}\, d\lambda; \tag{14_p}$$

(we have denoted by E_{ij} the $m \times m$ matrix with entries $E_{ij}^{rs} = \delta_{ir}\delta_{js}$). If we now fix all k_l, $l \neq p$, and apply the (1-dimensional) inhomogeneous Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{|z'|=R} \frac{f(z')}{z'-z}\, dz' + \frac{1}{2\pi i}\iint_{|z'|\leq R} \frac{\frac{\partial f}{\partial \bar{z}}(z')}{z'-z}\, dz' \wedge d\bar{z}' \tag{15}$$

to the variable k_p, we can convert (14_p) to an integral equation: noting that $\mu(x_0,x,k) \sim I$ when $|k_p| \to \infty$ [and denoting $k' = (k_1, \ldots, k'_p, \ldots, k_n)$], we have

$$\mu_\sigma(x_0,x,k) = I - \frac{i\bar{\sigma}}{|\sigma_I|(2\pi)^{n+1}} \sum_{i,j}\left(J_p^i - J_p^j\right) \iiint \frac{\delta\left(\sum J_l^i \lambda_l\right)}{k_p - k'_p}\, T_\sigma^{ij}(k',\lambda)$$

$$\times e^{i\beta_\sigma^{ij}(x_0,x,k',\lambda)} \mu_\sigma\left(x_0,x,\hat{k}^{ij}(k',\lambda)\right) E_{ij}\, d\lambda\, dk'_{R_p}\, dk'_{I_p}. \tag{16_p}$$

[More generally, one can use (15) with $f(z) = \mu_\sigma(x_0,x,k+zv)$, $z \in \mathbb{C}$, with k fixed and with an arbitrary $v \in \mathbb{C}^n$ which is not perpendicular to any of the vectors $J^i - J^j$, $i \neq j$.] The matrix-valued function $T_\sigma(k,\lambda)$ defined in (11) is our scattering data, and (16) is the inverse-scattering recipe for reconstructing μ from T. Once μ is found, the potential is easily recovered:

$$Q(x_0,x) = \frac{i\sigma}{\pi}\left[J_p, \iint \frac{\partial \mu_\sigma}{\partial \bar{k}_p}(x_0,x,k)\, dk_{R_p}\, dk_{I_p}\right]. \tag{17}$$

On the other hand, given an arbitrary $T(k,\lambda)$, to apply the above inversion procedure we would first need to know that the equations (14_p), $p = 1, 2, \ldots, n$,

are compatible; requiring that $\partial^2\mu/\partial\bar{k}_r\partial\bar{k}_p = \partial^2\mu/\partial\bar{k}_p\partial\bar{k}_r$ yields the following characterization equations for T:

$$\mathscr{L}^{ij}_{pr}[T_\sigma] \doteq \left(J^i_p - J^j_p\right)\frac{\partial T^{ij}_\sigma}{\partial \bar{k}_r} - \left(J^i_r - J^j_r\right)\frac{\partial T^{ij}_\sigma}{\partial \bar{k}_p}$$

$$+ \frac{i\bar{\sigma}}{2\sigma_l}\left(J^i_p - J^j_p\right)\left(J^i_r - J^j_r\right)\left(\frac{1}{J^i_r}\frac{\partial T^{ij}_\sigma}{\partial \lambda_r} - \frac{1}{J^i_p}\frac{\partial T^{ij}_\sigma}{\partial \lambda_p}\right)$$

$$= N^{ij}_{pr}[T_\sigma] \doteq \frac{i\bar{\sigma}}{2|\sigma_l|(2\pi)^n}\sum_{i'}\left[\left(J^{i'}_p - J^j_p\right)\left(J^i_r - J^{i'}_r\right) - \left(J^{i'}_r - J^j_r\right)\left(J^i_p - J^{i'}_p\right)\right]$$

$$\times \int \delta\left(\Sigma J^{i'}_l \lambda'_l\right) T^{i'j}_\sigma(k,\lambda') T^{ii'}_\sigma\left(\hat{k}^{i'j}(k,\lambda'), \lambda - \frac{J^{i'}}{J^i}\lambda'\right) d\lambda'. \quad (18^{ij}_{pr})$$

For compatibility, (18^{ij}) need only hold whenever $\Sigma J^i_l\lambda_l = 0$; however, one may also verify that T_σ when given by (11) satisfies (18) everywhere.

It turns out to be very useful to recast (18) in integral form. It is enough to keep only the equations (18_{p1}). We then look for a parametrization of the hyperplane $\{(k,\lambda)\in\mathbb{C}^n\times\mathbb{R}^n : \Sigma j^i_l\lambda_l = 0\}$ by new variables $(\chi, w_0, w)\in\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}^n$, so that, in the new coordinates, $\mathscr{L}^{ij}_{p1} = \partial/\partial\bar{\chi}_p$, $2\leqslant p\leqslant n$, and $\beta^{ij}_\sigma(x_0, x, k, \lambda) = x_0 w_0 + x\cdot w$; these requirements determine (up to a translation of χ) the following (invertible) map:

$$k_l = \left(J^j_l - J^i_1\right)\chi_l, \qquad l\geqslant 2,$$

$$k_1 = \sum_{l=2}^{n}\left(J^i_l - J^j_l\right)\chi_l + \frac{1}{J^j_1 - J^i_1}\left(\frac{\bar{\sigma}}{|\sigma|^2}w_0 + \sum_{l=1}^{n} J^i_l w_l\right),$$

$$\lambda_l = \frac{\left(J^j_1 - J^i_1\right)\left(J^i_l - J^j_l\right)}{\sigma_I J^i_l}\left(\sigma\chi_l\right)_I + w_l, \qquad l\geqslant 2,$$

$$\lambda_1 = \sum_{l=2}^{n}\left[\frac{\left(J^i_1 - J^j_1\right)\left(J^i_l - J^j_l\right)}{\sigma_I J^i_1}\left(\sigma\chi_l\right)_I - \frac{J^i_l}{J^i_1}w_l\right]. \quad (19)$$

To use (15) as before, we need the limit of T^{ij} for $|\chi_p|$ large (and χ_l, $l\neq p$, w_0, w fixed); this turns out to depend on whether for some $r\neq i, j$ we have

$$\left(J^r_1 - J^j_1\right)\left(J^i_p - J^j_p\right) = \left(J^i_1 - J^j_1\right)\left(J^r_p - J^j_p\right). \quad (20)$$

For brevity we consider only two cases [the only ones arising in the study of (2)—see the appendix]:

Case I. Equation (20) does *not* hold for any distinct i, j, r and any $p\neq 1$.

Case II. Equation (20) holds for all i, j, r, p (in other words, the vectors $J^1,\ldots,J^m$ all lie on the same line in $\mathbb{R}^n$).

In the generic case I we have

$$\lim_{|\chi_p| \to \infty} T_\sigma^{ij}(\chi, w_0, w) = \hat{Q}^{ij}(w_0, w). \tag{21}$$

and $(18_{p\mathrm{I}}^{ij})$ becomes

$$I_p^{ij}[T_\sigma](\chi, w_0, w) \doteq T_\sigma^{ij}(\chi, w_0, w) - \frac{1}{\pi}\iint \frac{N_{p\mathrm{I}}^{ij}[T_\sigma](\chi', w_0, w)}{\chi_p - \chi_p'}\, d\chi'_{R_p}\, d\chi'_{I_p}$$

$$= \hat{Q}^{ij}(w_0, w), \tag{22$_\mathrm{I}$}$$

where

$$\hat{Q}^{ij}(w_0, w) = \iint e^{-i(x_0 w_0 + x\cdot w)} Q^{ij}(x_0, x)\, dx_0\, dx \quad \text{and} \quad \chi' = (\chi_2, \ldots, \chi_p', \ldots, \chi_n).$$

If (20) holds for some $r \neq j$, then (21) need not be true [see (7), (8), (11)]. In case II we have $\partial T^{ij}/\partial \bar{\chi}_p \equiv 0$ for all p, $2 \leqslant p \leqslant n$; this, together with Liouville's theorem, allows us to replace (22_I) by

$$T_\sigma^{ij}(\chi, w_0, w) = T_\sigma^{ij}(w_0, w). \tag{22$_\mathrm{II}$}$$

In case I we conjecture (as in [2]) that the main condition needed to characterize the scattering data is that $I_p^{ij}[T_\sigma](\chi, w_0, w)$ are independent of χ and p and have suitable decay properties in (w_0, w); furthermore, given a T_σ which passes this admissibility test, we can (re)construct a local potential Q simply as the inverse Fourier transform of $I[T]$.

From (22_II) we see that T^{ij} is completely determined by its values on the $(n+1)$-dimensional surface $\chi = \chi_0$; the analogue of this in case I is the following: given $T_\sigma^{ij}(\chi_0, w_0, w) = G^{ij}(w_0, w)$, $1 \leqslant i, j \leqslant m$, we have [from (22_I)]

$$T_\sigma^{ij}(\chi, w_0, w) = G^{ij}(w_0, w)$$
$$+ \frac{1}{\pi}\iint \left[\frac{N_{p\mathrm{I}}^{ij}[T_\sigma](\chi', w_0, w)}{\chi_p - \chi_p'} - \frac{N_{p\mathrm{I}}^{ij}[T_\sigma](\chi_0', w_0, w)}{\chi_{0p} - \chi_p'}\right] d\chi'_{R_p}\, d\chi'_{I_p}. \tag{23}$$

which (at least for small G) could be solved to find T everywhere.

3. The case $\sigma = -1$.

If we formally substitute $\sigma = -1$ in (6), we find that, away from the hyperplanes $\Sigma_{ij} = \{k \in \mathbb{C}^n : \sum_{l=1}^n (J_l^i - J_l^j) k_{I_l} = 0\}$, the eigenfunction $\mu_{-1}(x_0, x, k)$ is well defined and holomorphic. Thus it appears that the inverse problem for the hyperbolic system L_{-1} could be regarded as a Riemann-Hilbert problem with data on the hyperplanes Σ_{ij}, $1 \leqslant i < j \leqslant m$. We prefer to obtain an inversion procedure from our results for $\sigma_I \neq 0$. There seems to be little advantage in studying the limit of $\mu_\sigma(x_0, x, k)$ as $\sigma \to -1$ (it leads us back to an analysis of singularities on the hyperplanes Σ_{ij}); we work instead with the limit of $\mu_\sigma(x_0, x, k_R, \sigma_I k_I)$, with k_I now playing the role of a parameter. From (6) we find

$$\lim_{\sigma \to -1+i0} G_\sigma(x_0, x, k_R, \sigma_I k_I) = G_L(x_0, x, k_R, k_I)$$

$$= \frac{-i}{(2\pi)^{n-1}} \iint_{\mathbb{R}^{n+1}} \left(\frac{\theta\left(\sum_{l=1}^n \left[J_l^i \xi_l + (k_{R_l} - k_{I_l})(J_l^i - J_l^j)\right]\right)}{\xi_0 - \sum_{l=1}^n \left[J_l^i \xi_l + k_{R_l}(J_l^i - J_l^j)\right] + i0} \right.$$

$$\left. + \frac{\theta\left(-\sum_{l=1}^n \left[J_l^i \xi_l + (k_{R_l} - k_{I_l})(J_l^i - J_l^j)\right]\right)}{\xi_0 - \sum_{l=1}^n \left[J_l^i \xi_l + k_{R_l}(J_l^i - J_l^j)\right] - i0} \right) e^{i(x_0 \xi_0 - x \cdot \xi)} \, d\xi_0 \, d\xi, \tag{24}$$

with $\theta(\cdot)$ the Heaviside function; correspondingly,

$$\lim_{\sigma \to -1+i0} \mu_\sigma(x_0, x, k_R, \sigma_I k_I) = \mu_L(x_0, x, k_R, k_I),$$

where μ_L solves the integral equation $\mu_L = I + \bar{G}_L(Q\mu_L)$. From (24) we see that $\mu_L(x_0, x, k_R, k_I)$ is a solution of

$$\frac{\partial \mu}{\partial x_0} - \sum_{l=1}^n J_l \frac{\partial \mu}{\partial x_l} - i \sum_{l=1}^n k_{R_l} [J_l, \mu] = Q\mu \tag{25}$$

for every value of the parameter k_I in $\mathbb{R}^n$. Our scattering data are now

$$T_L^{ij}(k_R, k_I, \lambda) = \iint_{\mathbb{R}^{n+1}} e^{-i\beta_L^{ij}(x_0, x, k_R, k_I, \lambda)} \left(Q(x_0, x)\mu_L(x_0, x, k_R, k_I)\right)^{ij} dx_0 \, dx \tag{26}$$

with

$$\beta_L^{ij}(x_0, x, k_R, k_I, \lambda)$$

$$= \sum_{l=1}^n \left\{ (J_l^i - J_l^j)\left[x_0 k_{I_l} - (x_l / J_l^i)(k_{R_l} - k_{I_l})\right] + (x_l + J_l^i x_0)\lambda_l \right\}.$$

Taking limits in (14), we find the reconstruction equation for μ:

$$\mu_L(x_0, x, k_R, k_I) = I + \frac{i}{(2\pi)^{n-1}} \sum_{i,j} \left(J_p^i - J_p^j \right)$$

$$\times \iiint \left[\frac{\theta\left(k_{I_p} - k'_{I_p}\right)}{k_{R_p} - k'_{R_p} + i0} - \frac{\theta\left(k'_{I_p} - k_{I_p}\right)}{k_{R_p} - k'_{R_p} - i0} \right] \delta\left(\sum J_l^i \lambda_l \right)$$

$$\times T_L^{ij}(k_R, k_I, \lambda) e^{i\beta_L^{ij}(x_0, x, k'_R, k'_I, \lambda)}$$

$$\times \mu_L\left(x_0, x, \hat{k}_L^{ij}(k'_R, k'_I, \lambda)\right) E_{ij}\, d\lambda\, dk'_{R_p}\, dk'_{I_p}. \tag{27}$$

where now

$$\left(\hat{k}_L^{ij}(k_R, k_I, \lambda)\right)_{R_l} = \frac{J_l^j}{J_l^i} k_{R_l} + \frac{J_l^i - J_l^j}{J_l^i} k_{I_l} + \lambda_l \quad \text{and} \quad \left(\hat{k}_L^{ij}\right)_{I_l} = k_{I_l}.$$

To write the characterization equations for T_L^{ij} we introduce new variables [suggested by the limit of (19)] $(\chi_R, \chi_I, w_0, w) \in \mathbb{R}^{3n-1}$ to parametrize the hyperplane $\sum J_l^i \lambda_l = 0$ in $\mathbb{R}^{3n}$ as follows:

$$k_{R_l} = \left(J_1^j - J_1^i \right) \chi_{R_l}, \qquad l \geqslant 2.$$

$$k_{R_1} = \sum_{l=2}^{n} \left(J_l^i - J_l^j \right) \chi_{R_l} + \frac{1}{J_1^i - J_1^j} \left(w_0 - \sum_{l=1}^{n} J_l^i w_l \right);$$

$$k_{I_l} = \left(J_1^j - J_1^i \right) \chi_{I_l}, \qquad l \geqslant 2;$$

$$k_{I_1} = \sum_{l=2}^{n} \left(J_l^i - J_l^j \right) \chi_{I_l} + \frac{1}{J_1^i - J_1^j} w_0;$$

$$\lambda_l = \frac{\left(J_1^j - J_1^i\right)\left(J_l^i - J_l^j\right)}{J_l^i} \left(\chi_{R_l} - \chi_{I_l} \right) + w_l, \qquad l \geqslant 2.$$

$$\lambda_1 = \sum_{l=2}^{n} \left[\frac{\left(J_1^i - J_1^j\right)\left(J_l^i - J_l^j\right)}{J_1^i} \left(\chi_{R_l} - \chi_{I_l} \right) - \frac{J_l^i}{J_1^i} w_l \right]. \tag{28}$$

Then under the assumption of case I in Section 2, the limit of the equations (22_I)

yields

$$T_L^{ij}(\chi_R, \chi_I, w_0, w) = \hat{Q}^{ij}(w_0, w) + \frac{1}{\pi} \iint \left[\frac{\theta(\chi_{I_2} - \chi'_{I_2})}{\chi_{R_2} - \chi'_{R_2} + i0} - \frac{\theta(\chi'_{I_2} - \chi_{I_2})}{\chi_{R_2} - \chi'_{R_2} - i0} \right]$$

$$\times N_{pl}^{ij}[T_L](\chi', w_0, w)\, d\chi'_{R_2}\, d\chi'_{I_2},$$

with $N_{pl}[T_L]$ given by a slight modification of (18). In case II we have

$$T_L^{ij}(\chi_R, \chi_I, w_0, w) = T_L^{ij}(w_0, w). \tag{29_{II}}$$

As in Section 2, we can now use (29_{I}) to characterize admissible T_L and (re)construct Q, as well as recover T_L from data given on $\chi_R = \text{const.}$ $\chi_I = \text{const.}$

It should be pointed out that once the family of Green's functions G_L has been chosen, all the above results can be derived without recourse to our limiting arguments [$\nabla_{k_I}\mu_L$ can be expressed in terms of μ_L using the appropriate symmetry property of G_L, and the analytic behavior of μ_L for k_I large—needed to establish (27)—follows from (24); these analytic properties, together with the compatibility requirements $\partial^2\mu/\partial k_{I_1}\partial k_{I_2} = \partial^2\mu/\partial k_{I_2}\partial k_{I_1}$, imply (29)]

4. Relation between T_L and the scattering operator ($\sigma = -1$)

To fix notation we sketch an elementary definition of the scattering operator associated with L_{-1}. When $Q \equiv 0$, given $f: \mathbf{R}^n \to \mathbf{R}^m$, the solution of the Cauchy problem $L_{-1}u(x_0, x) = 0$, $u(0, x) = f(x)$ is $u^i(x_0, x) = f^i(x_1 + x_0 J_1^i, \ldots, x_n + x_0 J_n^i)$, $1 \leqslant i \leqslant m$, which we write as $u(x_0, x) = f(x - x_0 J)$. When Q is, say, smooth and of compact support, given any (reasonable) $f: \mathbf{R}^n \to \mathbf{R}^m$, there is a unique u solution of $L_{-1}u = 0$ with $u(x_0, x) = f(x - x_0 J)$ for $x_0 \ll 0$; furthermore there is a unique g such that $u(x_0, x) = g(x - x_0 J)$ when $x_0 \gg 0$. We write $g = Sf$. On the Fourier transform side S can be written as

$$\hat{S}f(\xi) = f(\xi) + \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} S(\xi, k_R) \hat{f}(k_R)\, dk_R. \tag{30}$$

The question we would like to address is how to recover T_L (and hence Q) given $S(\xi, k_R)$. To relate T_L and $S(\xi, k_R)$ it turns out that we need to relate μ_L and the eigenfunction $\mu(x_0, x, k_R)$ corresponding to the "Volterra" Green's function

$$G^{ij}(x_0, x, k_R) = \theta(x_0) \exp\left[-i \sum_{l=1}^{n} (x_l - x_0 J_l^i) k_{R_l} \right] \prod_{l=1}^{n} \delta(x_l - x_0 J_l^i). \tag{31}$$

We start with the identity

$$\mu_L - \mu = (\bar{G}_L - \bar{G})(Q\mu_L) - \bar{G}(Q(\mu_L - \mu)). \tag{32}$$

write $G_L^{ij} - G^{ij}$ as a superposition of $\exp(i\beta_L^{ij})$, and use a suitable symmetry property of G. The main result is the following *linear* equation for T_L given S:

$$T_L^{ij}(k_R, k_I, \lambda) = S^{ij}(\hat{k}_R^{ij}(k_R, k_I, \lambda), k_R)$$
$$- \frac{1}{(2\pi)^n} \sum_{i'} \int_{\mathbf{R}^n} \theta\Big(\sum_{l=1}^{n} J_l^{i'} \lambda_l \Big)$$
$$\times S^{ii'}(\hat{k}_R^{ij}(k_R, k_I, \lambda), \hat{k}_R^{i'j}(k_R, k_I, \lambda')) T_L^{i'j}(k_R, k_I, \lambda')\, d\lambda', \tag{33}$$

where $\hat{k}_R^{ij}(k_R, k_I, \lambda)$ stands for the real part of $\hat{k}_L^{ij}$.

5. Applications to nonlinear equations

The equations (2) are the compatibility conditions (cf. [3]) for the Lax pair:

$$L_\sigma \psi = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial t} + \sum_{l=1}^{n} B_l \frac{\partial \psi}{\partial x_l} = A\psi; \tag{34}$$

the matrices B_l, $1 \leqslant l \leqslant n$, are constant real diagonal, and $A^{ij}(t, x_0, x) = (1/\sigma) a_{ij} Q^{ij}(t, x_0, x)$ with a_{ij} given by (3). The restrictions imposed by (3) on the matrices J_l, $1 \leqslant l \leqslant n$, are discussed in the appendix. To find the time dependence of the scattering data corresponding to (2), we set $\psi = \mu \exp[i\sum_{l=1}^{n} k_l(x_l - \sigma x_0 J_l - tB_l)]$; then μ satisfies (4) as well as

$$\mathscr{A}\mu \doteq \frac{\partial \mu}{\partial t} + \sum_{l=1}^{n} B_l \frac{\partial \mu}{\partial x_l} + i \sum_{l=1}^{n} k_l [B_l, \mu] - A\mu = 0. \tag{35}$$

Applying the operator $\mathscr{A}$ to both sides of the equation (14), we find (when $\sigma_I \neq 0$)

$$\frac{\partial T_\sigma^{ij}}{\partial t}(t, k, \lambda) = i \sum_{l=1}^{n} \left[B_l^j k_l - B_l^i \hat{k}_l^{ij}(k, \lambda) \right] T_\sigma^{ij}(t, k, \lambda). \tag{36}$$

For the case $\sigma = -1$ the equations [obtained as limits of (36) or by a parallel derivation] are

$$\frac{\partial T_L^{ij}}{\partial t}(t, k_R, k_I, \lambda) = i \sum_{l=1}^{n} \left[B_l^j k_{R_l} - B_l^i \hat{k}_{R_l}^{ij}(k_R, k_I, \lambda) \right] T_L^{ij}(t, k_R, k_I, \lambda). \tag{37}$$

Thus, when $\sigma = -1$, we can apply the inverse scattering procedure together with (37) to construct the solution to the Cauchy problem for (2). Note that

$T_L(t, k_R, k_I, \lambda)$ as given by (37) satisfies the characterization equations if $T_L(0, k_R, k_I, \lambda)$ does.

When $\sigma_I \neq 0$, the Cauchy problem for (2) is ill posed; however (by analogy to the corresponding linear problem) we can use inverse scattering to solve (2) as follows: given $Q(0, x_0, x)$, it can be decomposed into $Q_+(0, x_0, x) + Q_-(0, x_0, x)$, where $Q_+(0, x_0, x)$ extends to a function $Q_+(t, x_0, x)$ satisfying (2) in the half space $t > 0$, while $Q_-(0, x_0, x)$ extends to a function satisfying (2) in the half space $t < 0$. Assume for simplicity that $\sigma_I a_{ij} > 0$ for all $i \neq j$. Given Q, define $Q_\pm$ by $\hat{Q}_\pm(0, w_0, w) = \theta(\mp w_0)\hat{Q}(0, w_0, w)$. If T_+ is the scattering transform of Q_+, then from the direct problem we find $T_+^{ij}(0, \chi, w_0, w) = 0$ for $w_0 > 0$; thus for $t > 0$ we can define [see (36)] $T_+^{ij}(t, \chi, w_0, w)$ by

$$T_+^{ij}(t, \chi, w_0, w) = \exp\left[it \sum_{l=1}^{n} \left(B_l^i k_l - B_l^j \hat{k}_l^{ij} \right)\right] T_+^{ij}(0, \chi, w_0, w)$$

[see (3), (13), (19)]

$$= \exp\left[it\left(\frac{a_{ij}}{\sigma} w_0 + \sum_{l=1}^{n} \left(a_{ij} J_l^i - B_l^i \right) w_l \right)\right] T_+^{ij}(0, \chi, w_0, w). \tag{38}$$

Since the expression in the exponential does not depend on χ and since its real part is nonpositive if $t > 0$, $T_+^{ij}(t, \chi, w_0, w)$ satisfies the characterization equations (29), so inverse scattering should yield the desired potential $Q_+(t, x_0, x)$; similarly, we construct $Q_-(t, x_0, x)$, solution of (2) for $t < 0$.

Appendix

We need to find the restrictions imposed on the choice of matrices J_l, $1 \leq l \leq n$, by the existence of (a_{ij}) and B_l, $1 \leq l \leq n$, satisfying (3).

If (3) holds, then the matrix (a_{ij}) is symmetric and

$$a_{ip} - a_{pj} = (a_{ij} - a_{pj}) \frac{J_l^i - J_l^j}{J_l^p - J_l^j} \tag{A1}$$

for every l and every i, j, p distinct. [Conversely, if (A1) holds with (a_{ij}) symmetric, then B_l, $1 \leq l \leq n$, can be found so that (3) is satisfied.] Note that if $a_{ip} \neq a_{pj}$, (A1) forces J^i, J^j, J^p to be collinear. There are two cases:

I. *$a_{ip} = a_{pj}$ for all i, j, p distinct.* Then (A1) puts no restriction on J_l; in particular, they can be chosen so that (20) does not hold for any distinct i, j, r and $p \neq 1$. The system (2) is linear in this case.

II. *For some i_0, j_0, p_0 distinct, $a_{i_0 p_0} \neq a_{p_0 j_0}$.* We show that in this case the vectors $J^1, \ldots, J^m$ must *all* be collinear. From (A1) we already know that

$J^{i_0}, J^{j_0}, J^{p_0}$ are collinear. For any $r \neq i_0, j_0, p_0$ one of the following must be true:

$$\text{(i)}\ a_{i_0 r} \neq a_{r j_0}, \qquad \text{(ii)}\ a_{r i_0} \neq a_{i_0 p_0}, \qquad \text{(iii)}\ a_{r j_0} \neq a_{j_0 p_0} \tag{A2}$$

(for if not, $a_{i_0 p_0} = a_{r i_0} = a_{r j_0} = a_{p_0 j_0}$, contradicting our assumption). In any of the possibilities (A2), J^r will be on the line passing through $J^{i_0}, J^{p_0}, J^{j_0}$; this will be true for any r, $1 \leqslant r \leqslant m$. [Conversely, given $J^1, J^2, \ldots, J^m$ collinear with $J_l^i \neq J_l^j$, we can construct (a_{ij}) symmetric satisfying II and (A1).]

It follows that whenever (2) is not linear, the matrix having $J^1, J^2, \ldots, J^m$ as rows has rank at most 2; if $n \geqslant 3$, its columns [the diagonals of the matrices J_l in (1)] must be linearly dependent, and the inverse scattering problem for L_σ can also be solved by reducing it to a lower-dimensional one. On the other hand, since the characterization equations are trivial [i.e. $N(T) = 0$] in this case, it seems reasonable to expect that other (possibly nonlocal) nonlinear equations can be found which would be compatible with (22_{II}).

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The direct linearization of a class of nonlinear evolution equations

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The paper deals with the direct linearization, an approach used to generate particular solutions of the partial differential equations that can be solved through the inverse scattering transform. Linear integral equations are presented which enable one to find broad classes of solutions to certain nonlinear evolution equations in $1+1$ and $2+1$ dimensions.

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I. INTRODUCTION

The partial differential equations (PDE's) associated with the inverse scattering transform (IST) (see, for instance, Ref. 1 for details) are structurally rich. It is clear from the work done in this field that these equations admit many kinds of approaches and studies. Broadly speaking (see, for example, Ref. 2), it is possible to group these approaches in two different classes: "algebraic properties" and "methods of solution."

Among the algebraic properties one can associate with each of these PDE's are the existence of an infinite hierarchy of equations characterized by the same linear problem; the existence of infinitely many conserved quantities and of a Hamiltonian (sometimes bi-Hamiltonian) structure; the possibility of associating with these equations a so-called Bäcklund transformation (BT)—i.e., a nonlinear transformation connecting different solutions, etc.

The methods of solution developed so far depend of course on the specific problem that one has to solve: the IST for instance is the appropriate tool to solve the initial value problem associated with these PDE's.

In order to generate particular solutions there exist other methods: e.g., the BT; the Hirota approach[1]; the Dressing method[3]; and the Riemann–Hilbert direct approach,[4] introduced by Zakharov and Shabat (ZS); etc. The Dressing method has been formulated via an integral equation of the Gel'fand–Levitan–Marchenko (GLM) type, and the Riemann–Hilbert (RH) direct approach is based on a local homogeneous RH problem, used to generate solutions of the PDE. Later we will discuss in some detail the RH method, used often as a reference point of our analysis.

In this paper we will concentrate on a particular method of solution: the direct linearization (DL), an approach used to generate particular solutions of the PDE's that can be solved through the IST. We will (a) discuss earlier work and will give a natural generalization, which captures a significantly larger class of solutions; (b) stress the connections between this method and some of the main features of the IST; and (c) compare this linearization with the RH direct approach introduced by ZS, showing their connections and differences.

II. THE DIRECT LINEARIZATION

The DL was introduced by Fokas and Ablowitz[5] in connection with the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \quad u = u(x,t). \tag{1}$$

It is based essentially on the existence of an integral equation

$$\phi(x,t;k) + i\int_L \frac{\phi(x,t;l)}{l+k} e^{i(lx + l^3 t)}\, d\lambda(l) = 1, \tag{2}$$

involving an arbitrary contour L and measure $d\lambda(l)$ which linearizes Eq. (1). In fact, under the assumption that the homogeneous version of (2) has only the trivial solution, the solution ϕ of (2) provides a solution $u(x,t)$ of the KdV equation through the formula

$$u(x,t) = -\partial_x \int_L \phi(x,t;l) e^{i(lx + l^3 t)}\, d\lambda(l). \tag{3}$$

The original motivation for this result is associated with the, by now classical, IST (corresponding to $u \to 0$ sufficiently rapidly as $|x| \to \infty$) of the KdV equation. Specifically the integral equation (2), with contour and measure fixed and given in terms of the scattering data, is the integral formulation of the matrix RH problem,[2]

$$\begin{pmatrix} \bar\phi(x,t;k) \\ \phi(x,t;k) \end{pmatrix} = G(x,t;k) \begin{pmatrix} \bar\phi(x,t;-k) \\ \phi(x,t;-k) \end{pmatrix}. \tag{4}$$

In (4) ϕ is the same as in (2), $\bar\phi$ is another eigenfunction with appropriate analytic properties, and the matrix G is given in terms of the scattering data.

Another motivation is based on the Rosales perturbation approach[6]; in fact the solution (3) can be interpreted as the sum of the perturbation series solution of the KdV equation around the solution $u = 0$.

The arbitrariness of contour and measure in (2) allows one to capture a wider class of solutions than the one given by the GLM equation; as an example in Ref. 5 it was shown for instance that using (2) it is possible to find a three parameter family of solutions of the self-similar reduction of (1):

$$u''' - 6uu' - (2u + xu') = 0, \quad u = u(x). \tag{5}$$

The GLM equation is able to provide just one parameter family of solutions of (5).

Another suggestive argument is associated with the linear limit of (3); in this case, Eq. (3) becomes

$$u(x,t) = -\partial_x \int_L e^{i(kx + k^3 t)}\, d\lambda(k). \tag{6}$$

Equation (6) is *the general solution* ("Ehrenpreis principle") of the linearized KdV equation

$$u_t + u_{xxx} = 0. \tag{7}$$

The linear limit of (3) provides the most general solution of

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Eq. (7), whereas it is known that the linear limit of the GLM provides just those solutions of (7) obtainable using the Fourier transform.

These considerations are very far from implying that this DL provides the most general solution of (1); on the contrary recent studies on the equation of Painlevé II (PII),

$$v'' - xv - 2v^3 = \alpha, \tag{8}$$

which is intimately connected to Eq. (5) (see Ref. 7), have shown[3] that it is not the case, since the solutions of (5) obtained through (2) correspond just to the limited interval (0,1) of the parameter α in (8).

In other words, the perturbation solution (3) (in the Rosales language) of the KdV equation around $u = 0$ corresponds only to the solution of *PII in the interval* $0 < \alpha < 1$. *It* is natural to consider an extension of the DL formulated above which would correspond to the perturbation solution of the KdV equation around any arbitrary solution u_0 of the KdV itself.

III. A GENERALIZATION OF THE DL

The essence of this more general linearization is given by the following proposition.

Proposition 1: Let $\psi(x,t;k)$ be a solution of

$$\psi(x,t;k) + \int_L \psi(x,t;l)h(x,t;k,l)d\lambda(l) = \psi_0^{(1)}(x,t;k), \tag{9}$$

where L and $d\lambda(l)$ are arbitrary contour and measure; $\psi_0^{(1)}(x,t;k)$ and $\psi_0^{(2)}(x,t;k)$ are two arbitrary solutions of the coupled systems

$$\psi_{0_{xx}} + (u_0 + k^2/4)\psi_0 = 0, \tag{10a}$$

$$\psi_{0_t} = u_{0_x}\psi_0 + (k^2 - 2u_0)\psi_{0_x}; \tag{10b}$$

$u_0 = u_0(x,t)$ is any given solution of the KdV equation (1); and $h(x,t;k,l)$ is defined in terms of $\psi_0^{(1,2)}$ by

$$\begin{aligned} h(x,t;k,l) \doteq [2/(l^2 - k^2)] \big[&\psi_0^{(2)}(x,t;l)\psi_{0_x}^{(1)}(x,t;k) \\ &- \psi_{0_x}^{(2)}(x,t;l)\psi_0^{(1)}(x,t;k)\big]. \end{aligned} \tag{11}$$

Assuming that the homogeneous version of (9) has only the zero solution, then

$$u(x,t) = u_0(x,t) + \partial_x \int_L \psi(x,t;k)\psi_0^{(2)}(x,t;k)d\lambda(k) \tag{12}$$

solves the KdV equation.

The proof is direct and it is obtained operating on Eq. (9) with the operators P and $\mathcal{M}$ defined by

$$P \doteq \partial_{xx} + u + k^2/4, \quad \mathcal{M} \doteq -\partial_t + u_x + (k^2 - 2u)\partial_x. \tag{13}$$

The result of this operation gives

$$(P\psi)(x,t;k) + \int_L (P\psi)(x,t;l)h(x,t;k,l)d\lambda(l) = 0, \tag{14a}$$

$$\begin{aligned} (\mathcal{M}\psi)(x,t;k) &+ \int_L (\mathcal{M}\psi)(x,t;l)h(x,t;k,l)d\lambda(l) \\ &= \int_L (P\psi)(x,t;l)\psi_0^{(1)}(x,t;k)\psi_0^{(2)}(x,t;l)d\lambda(l), \end{aligned} \tag{14b}$$

and now if we assume that the homogeneous version of Eq. (9) has only the zero solution, Eq. (14a) implies that $P\psi = 0$ and then Eq. (14b) implies that $\mathcal{M}\psi = 0$. The compatibility between these two equations finally implies that u solves the KdV equation (1).

The linearization given here obviously contains the special cases in which $u_0 = 0$ and $u_0 = -2/x^2$, which are explicit solutions of Eq. (1); in these cases the DL was previously given.[5,9]

In the Appendix we give a constructive procedure that, starting with the general assumption (9), enables one to characterize the kernel h in terms of ψ_0 as in (11) and, at the same time, to fix the integral representation of $u - u_0$ in terms of ψ and ψ_0 as in (12). Such a systematic procedure, whose main steps are essentially the same for all the PDE's solvable via the IST, will be the basis of the results of this paper.

We remark that we could have given the DL of the KdV equation for the function $\phi(x,t;k) \doteq \psi(x,t;k)/\psi_0^{(1)}(x,t;k)$, instead of $\psi(x,t;k)$. In this case the corresponding integral equation

$$\phi(x,t;k) + \int_L \phi(x,t;l)g(x,t;k,l)d\lambda(l) = 1, \tag{15a}$$

$$g(x,t;k,l) \doteq \psi_0^{(1)}(x,t;l)h(x,t;k,l)/\psi_0^{(1)}(x,t;k) \tag{15b}$$

has 1 as forcing term and apparently would be the more appropriate formulation for investigating analyticity properties in k, in view of the solution of the IST. As far as the DL is concerned, the two formulations are completely equivalent and here and in the following we will use either one or the other, according to the convenience and to the elegance of the associated formulas.

The explicit formula (11) for the kernel h of Eq. (9) shows the singular character of the integral equation and strongly suggests that, as in the case $u_0 = 0$, some type of RH problem is going to be the natural structure underlying the IST of the KdV equation for solutions u, as a finite perturbation of a given solution u_0.

As we wrote above, the essence of this method is related to the existence of a linear integral equation like (9) [or (15a)] which provides solutions of the KdV equation. On the other hand, we know that the KdV equation is one of the many PDE's that can be solved through the IST. Hence it is natural to ask ourselves if and how the DL, in the generalized form introduced here, applies to other equations.

For this purpose, let us consider the $n \times n$ matrix equation

$$\Psi_x = zJ\Psi + Q\Psi, \quad \Psi = \Psi(x,t;k), \tag{16}$$

where the scalar constant z plays the role of spectral parameter, J is a constant diagonal matrix, and $Q = Q(x,t)$ is an off-diagonal matrix. Equation (16) is the natural $n \times n$ generalization of the generalized ZS problem (see Refs. 10 and 11) and its IST has been recently rigorously studied by Beals and Coifman.[12]

In order to give the DL associated with (16) it is convenient to introudce the matrix function $\Phi(x,t;k)$ defined by

$$\Phi(x,t;k) \doteq \Psi(x,t;k)\Psi_0^{(1)-1}(x,t;k), \tag{17}$$

where Ψ and $\Psi_0^{(1)}$ solve Eq. (16) corresponding to the two potentials Q and Q_0.

The linearization of the class of evolution equations associated with the spectral problem (16) is then given by the following.

Proposition 2: Let $\Phi(x,t;z)$ be a solution of

$$\Phi(x,t;z) - \int_L \Phi(x,t;l)G(x,t;z,l)d\lambda(l) = I, \tag{18}$$

when L and $d\lambda(l)$ are arbitrary contour and measure, I is the identity matrix, G is defined by

$$G(x,t;z,l) = (z-l)^{-1}\Psi_0^{(1)}(x,t;l) \times G_0(l)\Psi_0^{(2)-1}(x,t;l), \tag{19}$$

where $G_0(l)$ is an arbitrary constant matrix function, and the $\Psi_0^{(1,2)}$ are two arbitrary solutions of Eq. (16) for $Q_0(x,t)$. Assuming that the homogeneous version of (18) has only the trivial solution, then $\Psi(x,t)$ defined by

$$\Psi(x,t;z) = \Phi(x,t;z)\Psi_0^{(1)}(x,t;z) \tag{20}$$

solves Eq. (16) if

$$Q(x,t) = Q_0(x,t) + \left[J, \int_L \Phi(x,t;l)\Psi_0^{(1)}(x,t;l) \times G_0(l)\Psi_0^{(2)-1}(x,t;l)d\lambda(l)\right]. \tag{21}$$

In this proposition and in the following ones we often introduce arbitrary functions assuming that they satisfy suitable regularity properties in order to give sense to the integral formulas in which they appear.

Again the proof is direct and is obtained by applying the operator Ω,

$$(\Omega F)(x,t;z) \doteq -F_x - z[J,F] + QF - FQ_0, \tag{22}$$

on Eq. (18) to get

$$(\Omega\Phi)(x,t;z) - \int_L (\Omega\Phi)(x,t;l)G(x,t;z,l)d\lambda(l) = 0; \tag{23}$$

the result follows from the uniqueness assumption. In the Appendix we give a sketch of how to constructively obtain the linearization contained in Proposition 2, since the procedure does not differ in spirit from the one used for obtaining Proposition 1.

Problem (16) allows us to easily discuss the connections between the DL and the RH direct approach, indeed it will turn out that, if used on Eq. (16), then the two approaches are equivalent.

The RH direct approach introduced by ZS is based on the solution of the following matrix homogeneous RH problem:

$$\Phi^+(x,t;z) = \Phi^-(x,t;z)[I - R(x,t;z)], \tag{24}$$

where z lies on an arbitrary contour L in the complex-z plane, $\Phi^+(z)$ and $\Phi^-(z)$ are the boundary values on L of functions analytic inside and outside, respectively, of L, $\Phi^-(z) \to I$ as $|z| \to \infty$, and R is defined by

$$R(x,t;z) \doteq \Psi_0(x,t;z)G_0(z)\Psi_0^{-1}(x,t;z), \tag{25}$$

where $G_0(z)$ is an arbitrary constant matrix and Ψ_0 solves Eq. (16) with the potential Q_0. Then it is easy to verify [using (24), (25), and (16)] that $\Psi^\pm(x,t;z)$ and $Q(x,t)$, defined by

$$\Psi^\pm(x,t;z) = \Phi^\pm(x,t;z)\Psi_0(x,t;z), \tag{26}$$

$$zJ - Q(x,t) = [\Phi_x^\pm - \Phi^\pm(zJ - Q_0(x,t))](\Phi^\pm)^{-1}, \tag{27}$$

solve Eq. (16).

The equivalence between the DL given in Proposition 2 and the RH direct method is immediate and obtains by comparing (18) and (19) with the $(-)$ projection of (24):

$$\Phi^-(x,t;z) - \frac{1}{2\pi i}\int_L \Phi^-(x,t;l)\frac{R(x,t;l)}{l-z}dl = I, \tag{28}$$

where $z \to L$ from outside the contour.

The equivalence of the two approaches shows that the homogeneous RH problem (24) on which the ZS method is based, is the natural analytic structure underlying the linearization of the PDE's associated with the spectral pr[oblem] (16). The particular z-dependence of the kernel G of Eq. (18), given in (19), indicates that the integral equation (18) comes from the $(-)$ projection of a homogeneous RH problem of the type (24). Vice versa, if the z-dependence of G appeared in a different way, we would infer that (24) is not an adequate analytic structure for describing the problem. We will show in the following that this phenomenon is not exceptional, being a common feature of the PDE's related to the IST in $2+1$ dimensions.

While the RH approach (due to its restrictive basis) cannot in general be applied, the DL, based on a linear integral equation of the type (18), where the z-dependence of the kernel G is determined *a posteriori*, case by case [through direct algebraic calculations and is in general different from the one given in (19)], turns out to be a viable approach for characterizing a wide class of solutions of the PDE under investigation.

IV. THE DL IN 2 + 1 DIMENSIONS

The DL in $2+1$ dimensions is again based on a linear integral equation

$$\Phi(x,y,t;k) - \int_L \Phi(x,y,t;\beta(l,\nu)) \times G(x,y,t;k,l,\nu)d\zeta(l,\nu) = I. \tag{29}$$

Now the integration is in two variables l and ν, a reflection of the higher dimensionality of the configurational space. L and $d\zeta(l,\nu)$ are arbitrary contour and measure, $\beta = \beta(l,\nu)$ is an arbitrary function of l and ν, and the kernel G will be characterized in terms of certain linear PDE's whose coefficients are given in terms of the unperturbed solution $u_0(x,y,t)$.

As an example, let us apply the DL to the Kadomtsev–Petviashvili (KP) equation[13]

$$(u_t + 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy}, \quad \sigma \in \mathbb{C} \tag{30}$$

that can be obtained as a compatibility condition of the system

$$P\psi \doteq \sigma\psi_y + \psi_{xx} + u\psi = 0, \tag{31a}$$

$$M\psi \doteq \psi_t + 4\psi_{xxx} + 6u\psi_x + 3\left(u_x + \sigma\int_{-\infty}^{x} u_y(x')dx'\right)\psi = 0. \tag{31b}$$

In our analysis σ can be thought of as an arbitrary complex parameter, including then the two cases $\sigma = i$ and 1 (KPI and KPII) in which Eq. (30) describes the propagation of quasi-one-dimensional waves in a nonlinear weakly dispersive medium and the sign of σ^2 coincides with the sign of the dispersion.

We have the following proposition.

Proposition 3: Let $\psi(x,y,t;k)$ be a solution of

$$\psi(x,y,t;k) + \int_L \int \psi(x,y,t;\beta(l,\nu)) \times h(x,y,t;k,l,\nu) d\zeta(l,\nu) = \psi_0(x,y,t;k), \quad (32)$$

where ψ_0 solves the coupled system (31) corresponding to a given solution $u_0(x,y,t)$ of Eq. (30) and h is given by the formula

$$h(x,y,t;k,l,\nu) = \frac{1}{2}\int_\alpha^x f(x',y,t;l,\nu)\psi_0(x',y,t;k)dx' + \omega(y,t;k,l,\nu,\alpha), \quad (33)$$

where ω is a solution of the coupled system

$$\sigma\omega_y = \tfrac{1}{2}\left[f_x(\alpha)\psi_0(\alpha) - f(\alpha)\psi_{0_x}(\alpha)\right], \quad (34a)$$

$$\omega_t = -2\left[f_{xx}(\alpha)\psi_0(\alpha) - f_x(\alpha)\psi_{0_x}(\alpha) + f(\alpha)\psi_{0_{xx}}(\alpha)\right] - 3u_0 f(\alpha)\psi_0(\alpha), \quad (34b)$$

with

$$f(\alpha) \doteq f(\alpha,y,t;l,\nu), \quad \psi_0(\alpha) = \psi_0(\alpha,y,t;k),$$

and f solves

$$\sigma f_y - f_{xx} - u_0 f = 0, \quad (35)$$

$$f_t + 4f_{xxx} + 6u_0 f_x + 3\left(u_{0_x} + \sigma\int_{-\infty}^x u_{0_y} dx'\right) f = 0. \quad (36)$$

Assuming that the homogeneous version of (32) has only the trivial solution, then $u(x,y,t)$, defined through

$$u(x,y,t) = u_0(x,y,t) + \partial_x \int_L \int \psi(x,y,t;\beta(l,\nu)) \times f(x,y,t;l,\nu) d\zeta(l,\nu), \quad (37)$$

is a solution of the KP equation.

Again the proof is direct and it is based on the application of the operators P and $\mathcal{M}$ on Eq. (32). In the Appendix we show how the constructive procedure used to get Propositions 1 and 2 generalizes naturally to this $(2+1)$-dimensional example, hence enriching itself of new features and properties.

The solutions of the IST for KPI and KPII (see Refs. 14 and 15) can be easily recovered by choosing $u_0 = 0$ and $\beta = l$ for $\sigma = i$, and $u_0 = 0$ $\beta = l - i\nu$ for $\sigma = -1$.

The formulas (33) and (34) or, equivalently, the system of linear PDE's (A10) satisfied by h, have a rather complicated k-dependence. However, when $u_0 = 0$, the situation simplifies enormously; in order to see that, let us introduce the functions g and v defined as

$$g(x,y,t;k,l,\nu) \doteq h(x,y,t;k,l,\nu) \times \psi_0(x,y,t;\beta(l,\nu))/\psi_0(x,y,t;k), \quad (38a)$$

$$v(x,y,t;l,\nu) \doteq \psi_0(x,y,t;\beta(l,\nu)) f(x,y,t;l,\nu). \quad (38b)$$

Rewriting the system (A10) (including also the t-equation) for the function g, and considering the case $u_0 = 0$ (and $\psi_0(x,y,t;k) = \exp[ikx + (k^2/\sigma)y + 4ik^3 t]$), one obtains the overdetermined system

$$g_x + i(k-\beta)g = \tfrac{1}{2}v, \quad (39a)$$

$$\sigma g_y + (k^2 - \beta^2)g = \tfrac{1}{2}\left[v_x - i(k+\beta)v\right], \quad (39b)$$

$$g_t + 4i(k^3 - \beta^3)g = 2\left[(k^2 + \beta^2 + k\beta)v + i(k+2\beta)v_x - v_{xx}\right]. \quad (39c)$$

The compatibility condition for the system (39) fixes the k dependence of g in the form

$$g(x,y,t;k,l,\nu) = -iv(x,y,t;l,\nu)/[2(k - c(l,\nu))], \quad (40)$$

where $c = c(l,\nu)$ is an arbitrary function of l and ν and, correspondingly, v solves the equations

$$v_x = i(\beta - c)v, \quad (41a)$$

$$\sigma v_y = (\beta^2 - c^2)v, \quad (41b)$$

$$v_t = 4i(\beta^3 - c^3)v. \quad (41c)$$

The k-dependence of g (and then of h) implies that the integral equation (32) can be derived from a RH problem of the type introduced by Manakov[16] in a work in which he has generalized and adapted the RH direct approach of Ref. 4 to $2+1$ dimensions. He postulates a nonlocal RH problem,

$$\phi^+(x,y,t;k) = \phi^-(x,y,t;k) + \int \phi^-(x,y,t;l)G(x,y,t;k,l)dl, \quad (42)$$

in order to detect and generate PDE's solvable via the IST. The existence of explicit cases, associated with $u_0 = 0$ (and briefly discussed above), in which a RH structure is recovered, is a confirmation of the validity of Manakov's approach (for $u_0 = 0$) in finding a connection between the KP equation and the nonlocal RH problem (42). Such a connection was also proven via the solution of the IST (see Refs. 14, 15, and 17). In Ref. 15 in particular, for the first time it was shown that the KPII equation is related to a "$\bar\partial$" problem, whose integral representation also gives rise to the k-dependence presented in (40). But the nongenericity of the above-mentioned examples corresponding to the case $u_0 = 0$ indicates at the same time that the RH problem (42) is not adequate to capture a wide class of solutions of the KP equation.

We will show in the following that essentially the same situation arises when one writes the DL of a class of PDE's associated with the $2+1$ dimensional generalization of the spectral problem (16). Such a generalization is[18]

$$\Psi_x = J\Psi_y + Q\Psi, \quad (43)$$

where $\Psi = \Psi(x,y,t;k)$, $Q = Q(x,y,t)$ and J are $n\times n$ matrices, and J is constant and diagonal while Q is off diagonal. Physically relevant equations such as the so-called Davey–Stewartson equation, the n-wave interaction in $2+1$ dimensions, and the modified KP equation are related to (43). The IST associated with this linear problem has been recently investigated and solved in Refs. 19–21.

The DL corresponding to (43) is formulated in the following way.

Proposition 4: Let $\phi(x,y,k)$ be a solution of

$$\Phi(x,y,;k) + \int_L \int \Phi(x,y,;\beta(l,\nu)) \times G(x,y;k,l,\nu) d\zeta(l,\nu) = I, \quad (44)$$

where L and $d\zeta(l,\nu)$ are arbitrary contour and measure.

$\beta = \beta(l,\nu)$ is an arbitrary function of l and ν, G is given by the expression

$$G(x,y;k,l,\nu)$$
$$= \Psi_0(x,y;\beta(l,\nu))\Big(\int_\alpha^x \Psi_0^{-1}(x',y;\beta(l,\nu))$$
$$\times R(x',y;l,\nu)J\Psi_0(x',y,k)dx'$$
$$+ g(y,k,l,\nu,\alpha)\Big)\Psi_0^{-1}(x,y;k), \quad (45)$$

where

$$g_y(y,k,l,\nu,\alpha) = \Psi_0^{-1}(\alpha,y;\beta(l,\nu))R(\alpha,y;l,\nu)\Psi_0(\alpha,y;k), \quad (46)$$

$R = R(x,y;l,\nu)$ solves

$$-R_x + R_yJ + [J.S(x,y;\beta(l,\nu))R] + [Q_0(x,y),R] = 0, \quad (47)$$

with

$$S(x,y;\beta) = \Psi_{0_y}(x,y;\beta)\Psi_0^{-1}(x,y;\beta), \quad (48)$$

and Ψ_0 is a solution of (43) corresponding to the potential $Q_0(x,y)$. Then

$$\Psi(x,y;k) = \Phi(x,y;k)\Psi_0(x,y;k) \quad (49)$$

solves Eq. (43) if $Q(x,y)$ is given by

$$Q(x,y) = Q_0(x,y) + \Big[J,\int_L\int \Phi(x,y;\beta(l,\nu))$$
$$\times R(x,y;l,\nu)d\zeta(l,\nu)\Big]. \quad (50)$$

Again we refer to the Appendix for the derivation of this proposition. Formulas (45) and (46) imply that the kernel G satisfies the following set of (compatible) linear PDE's:

$$G_y + GS(k) - S(\beta)G = R, \quad (51a)$$

$$G_x + G(JS(k) + Q_0) - (JS(\beta) + Q_0)G = RJ. \quad (51b)$$

When $Q_0 = 0$ (and $\psi_0 = \exp[ik(Jx + y)]$), the compatibility condition for the system (51) fixes the k-dependence of G in the form

$$G(x,y;k,l,\nu) = -iR(x,y;l,\nu)/[k - c(l,\nu)], \quad (52)$$

and, correspondly, R is defined through the equations

$$R_x = i(\beta JR - cRJ), \quad (53a)$$

$$R_y = i(\beta - c)R \quad (53b)$$

postulated by Manakov in its nonlocal RH approach. This shows again how the nonlocal RH problem (42) is an appropriate tool to detect the PDE's in 2 + 1 dimensions corresponding to the linear problem (43), but, unlike the case of its associated 1 + 1 analog, it is able to capture a restricted class of solutions only (e.g., the ones obtained perturbing off of the zero solution).

Concluding this paper, we would like to remark that the DL has been studied here in connection with a certain selection of relevant eigenvalue problems associated with the IST theory, showing that the general assumptions on which the DL is based are consistent with the general features of the IST theory.

APPENDIX

In this Appendix we will illustrate the constructive procedure used in this paper in order to obtain the DL contained in Propositions 1–4. Since the main steps of this procedure are essentially the same for all of the PDE's related to the IST, we will discuss them in some detail for the KdV example, limiting our discussion of the other cases to those situations in which the procedure introduced needs to be modified or exhibits new features.

(1) The first step consists in writing the integral equation for ψ,

$$\psi(x;k) + \int_L \psi(x;l)h(x;k,l)d\lambda(l) = \psi_0^{(1)}(x;k), \quad (A1a)$$

or for $\phi = \psi\psi_0^{(1)-1}$,

$$\phi(x;k) + \int_L \phi(x;l)g(x;k,l)d\lambda(l) = 1. \quad (A1b)$$

The kernel h has to be characterized *a posteriori* as is indicated in the following steps.

(2) In the second step one applies the spectral operator P to (A1) [or Ω to (A2)]. In the KdV case $P = \partial_{xx} + u(x) + k^2/4$ and Eq. (A1a) implies

$$\psi_{xx}(x;k) + \int_L [\psi_{xx}(x;l)h(x;k,l) + 2\psi_x(x;l)h_x(x;k,l)$$
$$+ \psi(x;l)h_{xx}(x;k,l)]d\lambda(l) = \psi_{0_{xx}}(x;k), \quad (A2a)$$

$$u(x)\psi(x;k) + \int_L u(x)\psi(x;l)d\lambda(l)$$
$$= u_0(x)\psi_0(x;k) + [u(x) - u_0(x)]\psi_0(x;k), \quad (A2b)$$

$$\frac{k^2}{4}\psi(x;k) + \int_L\Big(\frac{l^2}{4}\psi(x;l)h(x;k,l)$$
$$+ \frac{k^2-l^2}{4}\psi(x;l)h(x;k,l)\Big)d\lambda(l) = \frac{k^2}{4}\psi_0(x;k). \quad (A2c)$$

Adding these three equations up, one obtains

$$(P\psi)(x;k) + \int_L (P\psi)(x;l)h(x;k,l)d\lambda(l)$$
$$+ \int_L \Big[2\psi_x(x;l)h_x(x;k,l) + \psi(x;l)$$
$$\times\Big(h_{xx}(x;k,l) + \frac{k^2-l^2}{4}h(x;k,l)\Big)\Big]d\lambda(l)$$
$$= (P_0\psi_0^{(1)})(x;k) + (u - u_0)\psi_0^{(1)}(x;k). \quad (A3)$$

Then the requirement $P\psi = 0$ ($P_0\psi_0$ is already zero by hypothesis) isolates an equation for $u - u_0$ which, in the KdV case, reads

$$(u - u_0)\psi_0^{(1)}(k) = 2\int_L \psi_x(l)f(k,l)d\lambda$$
$$+ \int_L \psi(l)\Big(h_{xx} + \frac{k^2-l^2}{4}h\Big)d\lambda \quad (A4)$$

here and below we omit for convenience the x dependence.

(3) The analysis of Eq. (A4) suggests the structure of the integral representation of $u - u_0$, in our case Eq. (A4) implies that $u - u_0$ *must* have the form

$$u - u_0 = \int_L [\psi_x(l)f_1(l) + \psi(l)f_2(l)]\,d\lambda, \tag{A5}$$

where the functions f_1 and f_2 are characterized in the next step.

(4) Evaluate the consequences of the assumed structure (A5). For instance in this specific case, Eqs. (A4) and (A5) imply

$$2h_x = f_1(l)\psi_0^{(1)}(k), \tag{A6a}$$

$$h_{xx} + [(k^2 - l^2)/4]h = f_2(l)\psi_0^{(1)}(k), \tag{A6b}$$

and one can verify that the compatibility condition for this system implies that $f_2 = f_{1_x} = \psi_{0_x}^{(2)}$, where $\psi_0^{(2)}$ is an arbitrary solution of the Schrödinger equation (10), and also that h is given by formula (11).

When applied to other examples, the procedure above repeats exactly for the first two steps, while steps 3 and 4 adjust to the specific problem under investigation. If, for instance, we deal with Eq. (16), after steps 1 and 2 we have

$$Q - Q_0 = \int_L \{(z - l)\,J\Phi G + \Phi\,[\,-G_x + (lJ + Q_0)G - G(zJ + Q_0)]\}\,d\lambda\,(l), \tag{A7}$$

and now taking into account that $Q - Q_0$ is a k-independent off-diagonal matrix, on analogy of (A5) we necessarily find the structure

$$Q - Q_0 = \left[\,J, \int_L \phi(l)R(l)\right]d\lambda, \tag{A8}$$

where again R has to be characterized. Substituting (A8) into (A7) we then obtain

$$(z - l)G(z,l) = R(l), \tag{A9a}$$

$$G_x + G(zJ + Q_0) - (lJ + Q_0)G = R(l)J. \tag{A9b}$$

System (A9) has the solution

$$G(k,l) = R(l)/(z - l), \quad R(l) = \psi_0^{(1)}(l)G_0(l)\psi_0^{(2)-1}(l).$$

The application of our procedure to equations in 2 + 1 dimensions leaves essentially unchanged the first three steps, and leads to the integral representations (A5) for KP and (A8) for the spectral problem (43). Equation (A5) implies for KP the following system:

$$2h_x = f_1(l)\psi_0(k), \tag{A10a}$$

$$\sigma h_y + h_{xx} = f_2(l)\psi_0(k), \tag{A10b}$$

whose compatibility condition implies that $f_{1_x} = f_2 = f_x$, and formula (34). Equation (A8) for the linear problem (43) implies the system (51), whose compatibility condition is given by Eq. (47).

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On the limit from the intermediate long wave equation to the Benjamin–Ono equation

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The intermediate long wave equation is a physically important singular integrodifferential equation containing a parameter, referred to here as δ. For $\delta \to \infty$ it reduces to the Benjamin–Ono equation. It has been recently shown that the inverse scattering transform schemes of the above equations have certain significant differences. Here it is shown that for $\delta \to \infty$, the inverse scattering transform scheme of the intermediate long wave equation reduces to that of the Benjamin–Ono equation.

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I. INTRODUCTION

The intermediate long wave (ILW) equation arises in the context of long internal gravity waves in a stratified fluid with finite depth.[1–6] Moreover, it arises in other circumstances as well (e.g., long waves in a stratified shear flow.[7,1] The ILW equation can be taken in the dimensionless form

$$u_t + (1/\delta)u_x + 2uu_x + Tu_{xx} = 0;$$

$$(Tv)(x) \doteq \frac{1}{2\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi(\xi - x)}{2\delta}\right)v(\xi)d\xi, \tag{1}$$

where Cauchy principal-value integrals are assumed if needed. In the internal gravity waves problem, the parameter δ can be thought as the ratio of depth to wavelength; Eq. (1) reduces to the Korteweg–deVries (KdV) equation[8] as $\delta \to 0$ (shallow-water limit)

$$u_t + 2uu_x + (\delta/3)u_{xxx} = 0, \tag{2}$$

and to the Benjamin–Ono (BO) equation[9] as $\delta \to \infty$ (deep-water limit)

$$u_t + 2uu_x + Hu_{xx} = 0; \quad (Hv)(x) \doteq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(\xi)}{\xi - x}d\xi. \tag{3}$$

Equations (1)–(3) are special cases of an equation discussed by Whitham.[10] *N-soliton solutions, an infinite number* of conserved quantities, Bäcklund transformations, and Lax pairs for the ILW and BO equations have been established in Refs. 3, 4, 6, 11, and in 12–16, respectively.

The inverse scattering transform (IST) scheme, a method for solving suitable initial-value problems for certain nonlinear equations, was discovered in connection with the KdV equation.[17] The IST schemes for the ILW and BO equations have been recently established in Refs. 11, 18, and 19, respectively. The limit of the IST scheme of the ILW equation to that of the KdV equation ($\delta \to 0$) is rather straightforward and was given in Refs. 6 and 18. However, the limit of the IST scheme of the ILW equation to that of the BO equation ($\delta \to \infty$) presents certain difficulties. This is a reflection of the fact that the IST schemes of the ILW and BO equations have significant differences. Actually the IST scheme of the ILW equation is conceptually similar to that of the KdV equation (see subsec. IIA below); on the other hand, the IST scheme of the BO equation is similar to that of the Kadomtsev–Petviashvili equation (a two-dimensional analog of the KdV equation).[20] Hence the limit process $\delta \to \infty$ in a sense provides a limit between two different types of IST formalisms, appropriate for one and two dimensional problems, respectively.

In this paper, it is established that as $\delta \to \infty$, the IST scheme of the ILW equation reduces to that of the BO equation.

II. REVIEW OF THE IST FOR THE ILW AND BO EQUATIONS

A. The ILW equation

The following results can be found in Ref. 18.

1. The direct scattering problem

The direct scattering problem of the ILW equation is based on the x-part (4a) of the "Lax pair"

$$L_\delta W \doteq iW_x^+ + [\zeta_+(\lambda) + 1/2\delta](W^+ - W^-) = -uW^+, \tag{4a}$$

$$iW_t^\pm - 2i\zeta_+ W_x^\pm + W_{xx}^\pm + [\mp iu_x - Tu_x + \rho_\delta]W^\pm = 0, \tag{4b}$$

where

$$\zeta_\pm(\lambda) \doteq \pm\frac{\lambda e^{\pm\lambda\delta}}{e^{\lambda\delta} - e^{-\lambda\delta}} \mp \frac{1}{2\delta}$$

$$\rho_\delta(\lambda) \doteq \lambda\zeta_+ + (\lambda/2)^2 + \nu, \ \nu$$

is an arbitrary constant, λ is a constant and is interpreted as a spectral parameter. Given u, Eq. (4a) defines a *Riemann–Hilbert problem in a strip of the complex x plane*; $W^\pm(x)$ represent the boundary values of functions [i.e., $\psi^\pm(x) = \lim_{y\to 0}\psi(x \pm iy)$] analytic in the horizontal strips between Im $z = 0$ and Im $z = \pm 2\delta$, and periodically extended vertically. Importantly, Eq. (4a) can also be solved

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without appealing to Riemann–Hilbert theory since it can be viewed as a differential-difference equation. This follows from the periodicity condition $\psi^-(x) = \psi^-(x + 2i\delta)$.

Let us concentrate on the $(+)$ functions and let $\mathcal{M}, \overline{\mathcal{M}}$ denote the $(+)$ "left" eigenfunctions, while $\overline{\mathcal{N}}, \mathcal{N}$ denote the $(-)$ "right" eigenfunctions. These eigenfunctions, in addition to solving (4), also satisfy the following boundary conditions

$$\mathcal{M} \to 1, \quad \overline{\mathcal{M}} \to e^{i\lambda x + \lambda\delta} \quad \text{as } x \to -\infty; \quad \overline{\mathcal{N}} \to 1,$$
$$\mathcal{N} \to e^{i\lambda x + \lambda\delta} \quad \text{as } x \to +\infty. \tag{5}$$

The eigenfunctions $\mathcal{M}, \overline{\mathcal{M}}, \mathcal{N}, \overline{\mathcal{N}}$ satisfy the following Fredholm integral equations

$$\begin{pmatrix} \mathcal{M}(x,\lambda) \\ \overline{\mathcal{M}}(x,\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\lambda x - \lambda\delta} \end{pmatrix} + \int_{-\infty}^{\infty} G_+(x,y,\zeta_+(\lambda)) u(y) \begin{pmatrix} \mathcal{M}(y,\lambda) \\ \overline{\mathcal{M}}(y,\lambda) \end{pmatrix} dy, \tag{6a}$$

$$\begin{pmatrix} \mathcal{N}(x,\lambda) \\ \overline{\mathcal{N}}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda x + \lambda\delta} \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} G_-(x,y,\zeta_+(\lambda)) u(y) \begin{pmatrix} \mathcal{N}(y,\lambda) \\ \overline{\mathcal{N}}(y,\lambda) \end{pmatrix} dy, \tag{6b}$$

where

$$G_\pm(x,y,\zeta_+(\lambda)) \doteq \frac{1}{2\pi} \int_{C_\pm} dp \, \frac{e^{ip(x-y)}}{p - [\zeta_+(\lambda) + 1/2\delta](1 - e^{-2p\delta})}, \tag{7}$$

where $C_\pm$ are the contours $\operatorname{Re}(p \mp i0)$.

The eigenfunctions $\mathcal{M}, \overline{\mathcal{N}}, \mathcal{N}$ are related through the scattering equation

$$\mathcal{M}(x,\lambda) = a(\lambda)\overline{\mathcal{N}}(x,\lambda) + \theta(\zeta_+(\lambda) + 1/2\delta) b(\lambda)\mathcal{N}(x,\lambda), \tag{8}$$

where

$$a(\lambda) \doteq 1 + \frac{1}{2i\zeta_+(\lambda)} \int_{-\infty}^{\infty} dy\, u(y)\mathcal{M}(y,\lambda);$$

$$b(\lambda) \doteq -\frac{1}{2i\zeta_-(\lambda)\delta} \int_{-\infty}^{\infty} dy\, u(y)\mathcal{M}(y\lambda) e^{-i\lambda y - \lambda\delta} \tag{9}$$

and

$$\theta(\lambda) = 1 \quad \text{for } \lambda > 0, \quad \theta(\lambda) = 0 \quad \text{for } \lambda < 0.$$

The "bound states" correspond to those $\hat{\lambda}_l$ for which

$$a_l \doteq a(\hat{\lambda}_l) = 0, \quad l = 1,2,\ldots,n, \tag{10a}$$

$$\mathcal{M}_l(x) \doteq \mathcal{M}(x\hat{\lambda}_l) = b(\hat{\lambda}_l)\mathcal{N}(x\hat{\lambda}_l) \doteq b_l \mathcal{N}_l(x). \tag{10b}$$

2. The inverse scattering problem

The solution of the inverse problem is based on Eq. (8). Given $a(\lambda)$, $b(\lambda)$, and appropriate information about the bound states, find $\mathcal{M}, \overline{\mathcal{N}}, \mathcal{N}$. In order to view (8) as a *Riemann–Hilbert problem in the complex $\zeta_+(\lambda)$ plane*, one needs to establish analyticity properties in $\zeta_+(\lambda)$ for the eigenfunctions $\mathcal{M}, \overline{\mathcal{N}}, \mathcal{N}$. The Kernels of the integral equations satisfied by $\mathcal{M}, \overline{\mathcal{N}}$ are $(+)$ and $(-)$ functions, respectively, in $\zeta_+(\lambda)$, i.e., they are analytically extendable in the appropriate regions of the $\zeta_+(\lambda)$ plane. Since the forcing in both cases is unity, Fredholm theory implies that the solutions $\mathcal{M}$ and $\overline{\mathcal{N}}$ are also $(+)$ and $(-)$ functions in $\zeta_+(\lambda)$, provided that there exist no solutions to Eqs. (6) (when δ is finite, it can be shown that for suitable potentials this is actually the case). Furthermore, Eq. (9a) implies that $a(\zeta_+)$ is a $(+)$ function in λ.

In order to solve (8), one needs to establish an *analytic connection* or symmetry condition between $\overline{\mathcal{N}}$ and $\mathcal{N}$. This follows from the relationship

$$\mathcal{N}(x,\lambda) = \overline{\mathcal{N}}(x, -\lambda) e^{i\lambda x - \lambda\delta}, \tag{11}$$

which is a consequence of

$$G_\pm(x,y,\lambda) = G_\pm(x,y,-\lambda) e^{i\lambda(x-y)}. \tag{12}$$

Equation (8), using the above analytic properties of $\mathcal{M}$, $\overline{\mathcal{N}}$, and a, as well as Eq. (11), defines a Riemann–Hilbert problem in $\zeta_+(\lambda)$. From this, the following integral equation is obtained (see Appendix A):

$$\overline{\mathcal{N}}(\zeta_+) - \frac{1}{2\pi i} \int_{-1/2\delta}^{\infty} \frac{\rho(\zeta'_+)\mathcal{N}(\zeta'_+)}{\zeta'_+ - (\zeta_+ - i0)} d\zeta'_+ = 1 + i\sum_{j=1}^{n} \frac{C_j \mathcal{N}_j}{\zeta_+ - \zeta_{+j}}, \tag{13}$$

where

$$C_j \doteq -i \frac{b(\lambda)}{a_{\zeta_+}(\zeta_+(\lambda))}\bigg|_{\lambda = \hat{\lambda}_j}, \quad \rho \doteq \frac{b(\lambda)}{a(\lambda)}. \tag{14}$$

The Gel'fand–Levitan–Marcenko equation given in Ref. 18 can be easily obtained by taking an appropriate Fourier transform of (13), supplied by the analytic information (11).

We shall also need the following relationship, which is obtained from (13) asymptotically as $\zeta_+ \to \infty$ (see Appendix A):

$$u^+(x) = \frac{1}{2\pi i} \int_{-1/2\delta}^{\infty} \rho(\zeta_+)\mathcal{N}(\zeta_+) d\zeta_+ - i\sum_{j=1}^{n} C_j \mathcal{N}_j;$$

$$u^+(x) \doteq \frac{1}{4i\delta} \int_{-\infty}^{\infty} \cosh\left(\frac{\pi(y - x - i0)}{2\delta}\right) u(y) dy. \tag{15}$$

And finally, the reality of $u(x)$ implies that $u(x) = u^+(x) - u^-(x) = u^+(x) + (u^+(x))^*$.

Equation (13) defines $\mathcal{N}$ in terms of ρ, C_j, $\hat{\lambda}_j$, and Eq. (15) defines u^+ in terms of ρ, C_j, $\mathcal{N}$. Hence Eqs. (13) and (15) define $u^+(x)$ in terms of ρ, C_j, $\hat{\lambda}_j$, the so-called scattering data. However, the scattering data need only be evaluated at time $t = 0$ [i.e., in terms of the initial data $u(x,0)$ only] since their evolution is known.

$$\{\hat{\lambda}_j(t) = \hat{\lambda}_j(0),$$
$$C_j(t) = C_j(0)\exp(i\hat{\lambda}_j[\hat{\lambda}_j \coth(\hat{\lambda}_j\delta) - 1/\delta]t)\}_{j=1}^{n},$$
$$\rho(\lambda,t) = \rho(\lambda,0)\exp[i\lambda(\lambda\coth\lambda\delta - 1/\delta)t]. \tag{16}$$

The above evolution of the scattering data follows easily from the t-part (4b) of the Lax pair.

B. The BO equation

The following results can be found in Ref. 19.

1. The direct scattering problem

The direct scattering problem of the BO equation is based on the x part (17a) of the "Lax pair":

$$Lw \doteq iw_x^- + \lambda(w^+ - w^-) = -uw^+, \quad (17a)$$

$$iw_t^\pm - 2i\lambda w_x^\pm - w_{xx}^\pm - 2i[u]_x^\pm w^\pm = -\rho w^\pm, \quad (17b)$$

where

$$[u]^\pm \doteq \pm u/2 + (1/2i)Hu. \quad (17c)$$

Given u, Eq. (17a) defines a differential *Riemann–Hilbert problem in the complex x-plane*; $w^\pm(x)$ represent the boundary values of functions analytic in the upper (+) and lower (−) half x-plane, i.e., $w^\pm(x) = \lim_{y\to 0} w(x \pm iy)$.

The eigenfunctions $m, \bar{m}, n, \bar{n}$ satisfy the boundary conditions

$$m \to 1, \quad \bar{m} \to e^{i\lambda x} \quad \text{as } x \to -\infty;$$
$$\bar{n} \to 1, \quad n \to e^{i\lambda x} \quad \text{as } x \to +\infty, \quad (18)$$

and are characterized through the following Fredholm integral equations:

$$\begin{pmatrix} m(x,\lambda) \\ \bar{m}(x,\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\lambda x} \end{pmatrix} + \int_{-\infty}^{\infty} g_+(x,y,\lambda)u(y)\begin{pmatrix} m(y,\lambda) \\ \bar{m}(y,\lambda) \end{pmatrix} dy, \quad (19a)$$

$$\begin{pmatrix} n(x,\lambda) \\ \bar{n}(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\lambda x} \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} g_-(x,y,\lambda)u(y)\begin{pmatrix} n(y,\lambda) \\ \bar{n}(y,\lambda) \end{pmatrix} dy, \quad (19b)$$

where g_+, g_- are the (+) and (−) parts of sectionally holomorphic function

$$g(x,y,\Lambda) = \frac{1}{2\pi}\int_0^\infty dp\,\frac{e^{i(x-y)p}}{p-\Lambda}, \quad (20)$$

and Λ denotes the complex extension of λ, i.e.,

$$g_\pm(x,y,\lambda) = \frac{1}{2\pi}\int_0^\infty dp\,\frac{e^{i(x-y)p}}{p-(\lambda \pm i0)}. \quad (21)$$

The eigenfunctions $m, \bar{n}, n$ are related through the scattering equation

$$m(x,\lambda) = \bar{n}(x,\lambda) + \theta(\lambda)\beta(\lambda)n(x,\lambda), \quad (22)$$

where

$$\beta(\lambda) = i\int_{-\infty}^{\infty} u(y)m(y\lambda)e^{-i\lambda y}dy. \quad (23)$$

2. The inverse scattering problem

The solitons of the ILW equation correspond to "bound states" which are generated from the zeros of $a(\lambda)$. However, in Eq. (22) [which is the analog of (8)], the coefficient of $\bar{n}$ is 1. Hence the solitons of the BO equation are generated through a different mechanism: the integral equations (19), in contrast to the integral equations (6), may have homogeneous solutions Φ_j, for some λ_j, where $\lambda_j < 0$, i.e.,

$$\Phi_j(x) = \int_{-\infty}^{\infty} g(x,y\lambda_j)u(y)\Phi_j(y)dy, \quad \lambda_j < 0. \quad (24)$$

The kernels of the integral equations for $m, \bar{n}$ are (+) and (−) functions respectively in λ. Hence

$$m(x,\lambda) = 1 + \sum_1^n \frac{C_j\Phi_j(x)}{\lambda-\lambda_j} + m_-(x,\lambda), \quad (25)$$

$$\bar{n}(x,\lambda) = 1 + \sum_1^n \frac{\bar{C}_j\Phi_j(x)}{\lambda-\lambda_j} + \bar{n}_-(x,\lambda), \quad (26)$$

where $m_-, \bar{n}_-$ are (+) and (−) functions, respectively, in λ. It turns out that

$$C_j = \bar{C}_j = -i, \; j = 1,2,\dots,n.$$

In order to view Eq. (22) as a Riemann–Hilbert problem in the complex λ plane, one needs to establish analytic information about n and $\bar{n}$. This follows from

$$\frac{\partial}{\partial\lambda}(n(x,\lambda)e^{-i\lambda x}) = f(\lambda,t)e^{-i\lambda x}\bar{n}(x\lambda);$$

$$f(\lambda) \doteq -\frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} u(y)n(y\lambda)dy. \quad (27)$$

Equation (27) is a consequence of

$$\frac{\partial}{\partial\lambda} g_\pm(x,y,\lambda) = -\frac{1}{2\pi\lambda} + i(x-y)g_\pm(x,y,\lambda). \quad (28)$$

Using (28), one also finds that

$$\lim_{\lambda\to\lambda_j}[\bar{n}(x,\lambda) - i\Phi_j(x)/(\lambda-\lambda_j)] = (x+\gamma_j)\Phi_j. \quad (29)$$

Equation (22), using (25)–(27), and (29), defines a *nonlocal Riemann–Hilbert problem in the complex λ plane* which is equivalent to the integral equation

$$n(x,\lambda,t) = \frac{1}{2\pi}\int_0^\infty h(x,t,\lambda,l)\beta(l,t)n(x,t,l)dl + \sum_{l=1}^n \Phi_l(x,t)h(x,t,\lambda,\lambda_l) = v(x,t,\lambda), \quad (30)$$

$$(x+\gamma_j(t))\Phi_j(x,t) - \frac{1}{2\pi i}\int_0^\infty \frac{\beta(l,t)n(x,t,l)}{\lambda-\lambda_j}dl + i\sum_{\substack{l=1 \\ l\neq j}}^n \frac{\Phi_l(x)}{\lambda_j-\lambda_l} = 1, \quad (31)$$

where

$$v(x,t,\lambda) \doteq \int_0^\lambda (f(l,t)e^{i\alpha(\lambda-l)} + f_s(l)e^{i\alpha\lambda})dl; \quad f_s(\lambda) = \frac{1}{\lambda\ln\lambda}, \quad (32a)$$

$$h(x,t,\lambda,l) \doteq e^{i(\lambda-l)x}\int_\infty^x v(\xi,\lambda)e^{-i(\lambda-l)\xi}d\xi, \quad l>0, \quad (32b)$$

$$h(x,t,\lambda,\lambda_j) \doteq e^{i(\lambda-l)x}\int_\alpha^x v(\xi,\lambda)e^{-i(\lambda-\lambda_j)\xi}d\xi + e^{i(\lambda-\lambda_j)(x-\alpha)}\int_0^\lambda \left(\frac{f(l,t)e^{i\alpha(\lambda-l)}}{\lambda_j-l} + \frac{f_s(l)}{\lambda_j}\right)dl. \quad (32c)$$

The following equation is also valid:

$$[u]^+ = \frac{1}{2\pi i}\int_0^\infty \beta(\lambda)n(x\lambda)d\lambda + i\sum_1^n \Phi_j(x), \quad (33)$$

and assuming u real, $u(x) = u^+(x) + (u^+(x))^*$.

Equations (30)–(33) define $[u]^+$ in terms of $\lambda_j, \gamma_j, \beta(\lambda)$, $f(\lambda)$. However, the scattering data need only be evaluated at time $t = 0$, since their evolution is known from (17b):

$$\{\lambda_j(t) = \lambda_j(0), \quad \gamma_j(t) = 2\lambda_j t + \gamma_j(0)\}_{j=1}^{n},$$
$$\beta(\lambda, t) = \beta(\lambda, 0)e^{i\lambda^2 t}, \quad f(\lambda, t) = f(\lambda, 0)e^{i\lambda^2 t}. \tag{34}$$

III. THE LIMIT FROM THE ILW EQUATION TO THE BO EQUATION

In this section, we will show how the IST scheme of the BO equation can be obtained from the IST scheme of the ILW equation.

A. The direct scattering problem

As noticed in Ref. 18, the limit $\delta \to \infty$, $\lambda > 0$, of the Lax pair (4) goes directly to the system (19); the strips between $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pm 2\delta$ become the upper and lower half z-plane, and then $w^{\pm}(x) = \lim_{\delta\to\infty} W^{\pm}(x)$ are nothing but the boundary values of functions analytic in the upper $(+)$ and lower $(-)$ half z-plane.

It is straightforward to show [see (B2)] that $\lim_{\delta\to\infty} G_{\pm}(x, y, \zeta_{+}(\lambda)) = g_{\pm}(x, y, \lambda)$, $\lambda > 0$, where the Green functions $G_{\pm}$ and $g_{\pm}$ are defined in (7) and (21), respectively, then the Jost functions of (17a) are solutions of the Fredholm equations (19) and can be obtained, for $\lambda > 0$, in the following way:

$$m(x, \lambda) = \lim_{\delta\to\infty} M(x, \lambda) \tag{35a}$$

$$\bar{m}(x, \lambda) = \lim_{\delta\to\infty} \bar{M}(x, \lambda)e^{-\lambda\delta}, \tag{35b}$$

$$n(x, \lambda) = \lim_{\delta\to\infty} N(x, \lambda)e^{-\lambda\delta}, \tag{35c}$$

$$\bar{n}(x, \lambda) = \lim_{\delta\to\infty} \bar{N}(x, \lambda). \tag{35d}$$

The analytic information about $G_{\pm}$ (and, consequently, about N) contained in (11) and (12) are apparently lost in the limit $\delta \to \infty$, from which we find the identity $g_{\mp} = g_{\pm}$. Nevertheless, one may show that taking the derivative of (12) with respect to λ,

$$G_{\pm\lambda}(x, y, \lambda) = i(x - y)G(x, y, \lambda) + G_{\lambda}(x, y, -\lambda)e^{i\lambda(x-y)}, \tag{36}$$

and *then* taking the limit $\delta \to \infty$ of this equation, one gets [see, (B3)–(B5)] the nontrivial equation (28). Analogously, by taking the λ derivative of Eq. (6b) for $\bar{N}(x, -\lambda)$, enriched by the property (11),

$$(N(x, \lambda)e^{-i\lambda x - \lambda\delta})_{\lambda}$$
$$= \int_{-\infty}^{+\infty} G_{\lambda}(x - y, -\lambda)u(y)N(y, \lambda)e^{-i\lambda y - \lambda\delta}dy$$
$$+ \int_{-\infty}^{+\infty} G(x - y, \lambda)e^{i\lambda(x-y)}u(y)(N(y, \lambda)e^{-i\lambda y - \lambda\delta})_{\lambda}dy, \tag{37}$$

and *then* taking the limit $\delta \to \infty$, one gets the analytic connection formula (27). This highly nontrivial formula is derived at this stage as a consequence of the noncommutativity of the two operators $\lim_{\delta\to\infty}$ and $\partial/\partial\lambda$. It will be rederived later (perhaps in a more satisfactory way) from the scattering problem.

Using Eqs. (35) for $\lambda > 0$, together with the symmetry condition (11) for the case $\lambda < 0$, one can take the limit of the scattering equation (8). Specifically,

$$\lim_{\delta\to\infty} a(\lambda) = \begin{cases} 1, & \lambda > 0, \quad (38a) \\ d(\lambda), & \lambda < 0, \quad (38b) \end{cases}$$

$$\lim_{\delta\to\infty} b(\lambda) = e^{-\lambda\delta}\beta(\lambda), \quad \lambda > 0, \tag{39a}$$

$$\lim_{\delta\to\infty} \delta e^{\lambda\delta} b(\lambda) = \frac{i}{2\lambda}\int_{-\infty}^{\infty} u(y)\bar{m}(y, -\lambda)dy, \quad \lambda < 0, \tag{39b}$$

where $d(\lambda) \doteq 1 + i\int_{-\infty}^{\infty} u(y)\bar{m}(y, -\lambda)e^{i\lambda y}dy$, and $\beta(\lambda)$ is defined in (23). Then in the limit $\delta\to\infty$, Eq. (8) goes to Eq. (22) for $\lambda > 0$, and it goes to

$$\bar{m}(x, -\lambda) = d(\lambda)n(x, -\lambda) \tag{40}$$

for $\lambda < 0$.

Finally, (38) and (39), together with (14) and (40), imply that

$$\lim_{\delta\to\infty} \rho(\lambda) = e^{-\lambda\delta}\beta(\lambda), \quad \lambda > 0, \tag{41a}$$

$$\lim_{\delta\to\infty} \delta e^{\lambda\delta}\rho(\lambda) = i\pi f(-\lambda), \quad \lambda < 0, \tag{41b}$$

with $f(\lambda)$ defined in (27).

The solution of the inverse problem for the BO equation will be obtained taking the limit $\delta\to\infty$ of Eq. (13). However, in order to do that, we must still characterize the asymptotics of the bound states $\hat{\lambda}_j = \hat{\lambda}_j(\delta)$, $j = 1,2,...,n$ of the ILW *equation.*

B. The bound states

As shown in Ref. 18 for every finite δ the $\hat{\lambda}_j$'s are simple zeros of $a(\lambda)$ and lie on that portion of the imaginary axes contained in the fundamental sheet of the λ plane: $\hat{\lambda}_j = ik_j$, $0 < k_j < \pi/\delta$.

In order to establish the asymptotics of $\hat{\lambda}_j$, we will study the equation $a_j \doteq a(\hat{\lambda}_j, \delta) = 0$ for large δ with the following ansatz:

$$\hat{\lambda}_j = (i\pi/\delta)(\alpha_0^{(j)} + \alpha_1^{(j)}/\delta + \alpha_2^{(j)}/\delta^2 + O(\delta^{-3})), \tag{42}$$

and the restriction $0<\alpha_0^{(j)}<1$, which is a direct consequence of the property $0 < k_j < \pi/\delta$.

Substituting ansatz (42) into the equation $a_j = 0$ evaluated for large δ, one gets an equation in inverse powers of δ. In order to equate to zero the coefficients of the $O(1)$ term, the following conditions must be satisfied (see Appendix C):

$$\alpha_0^{(j)} = 1, \tag{43}$$

$$M_j(x) = \delta\mu_0^{(j)}(x) + \mu_1^{(j)}(x) + O(\delta^{-1}), \tag{44}$$

$$\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)dy = -2i\lambda_j, \tag{45a}$$

where

$$\lambda_j \doteq 1/2\alpha_1^{(j)}; \tag{45b}$$

while, equating to zero the coefficient of the $O(\delta^{-1})$ term, one gets

$$\int_{-\infty}^{\infty} u(y)\mu_1^{(j)}(y)dy = -2i\nu_{-,j}, \tag{46a}$$

where

$$\nu_{\mp,j} = i(\pi/2) \mp 2\lambda_j^2 \alpha_2^{(j)}. \tag{46b}$$

As a consequence of result (43), the property $k_j < \pi/\delta$ implies that $\alpha_1^{(j)}$ (and then λ_j) is negative, otherwise arbitrary.

Moreover

$$\zeta_{\pm,j} = \zeta_{\pm}(\hat{\lambda}_j) = \pm\lambda_j + \nu_{\pm,j}/\delta + O(\delta^{-2}), \quad \delta \gg 1. \tag{47}$$

Substituting the expansion (44) into Eq. (6a) evaluated for large δ and equating to zero the coefficients of the first two terms, we get the integral equations satisfied by $\mu_0^{(j)}(x)$ and $\mu_1^{(j)}(x)$:

$$(K\mu_0^{(j)})(x) = 0, \tag{48a}$$

$$(K\mu_1^{(j)})(x) = 1 + \int_{-\infty}^{\infty} g_1(x, y, \lambda_j)u(y)\mu_0^{(j)}(y)dy, \tag{48b}$$

where

$$(K\eta)(x) = \eta(x) - \int_{-\infty}^{\infty} g(x, y, \lambda_j)u(y)\eta(y)dy, \tag{49a}$$

and

$$g_1(x, y, \lambda_j) = (\nu_{-,j} + \tfrac{1}{2})g_\lambda(x, y, \lambda_j) - i/4\lambda_j \tag{49b}$$

is the coefficient of the $O(\delta^{-1})$ term in the expansion of $G_-(x, y, \hat{\lambda}_j)$ when $\delta \gg 1$ [see (B6)]:

$$G_-(x, y, \hat{\lambda}_j) = g(x, y, \lambda_j) + (1/\delta)g_1(x, y, \lambda_j) + O(\delta^{-2}). \tag{49c}$$

Equation (48a) shows that the leading term of the expansion of $\mathcal{M}_j$ is a solution of the homogeneous equation (24), corresponding to the eigenvalue λ_j. $\mu_1^{(j)}(x)$ is the solution of the inhomogeneous equation (48b) and the necessary and sufficient condition for such a solution to exist in [see (D5)] $\alpha_2^{(j)} = 1/(4\lambda_j^2)$, whereupon then $\nu_{\mp,j} = \tfrac{1}{2}(i\pi \mp 1)$. So Eq. (48b) becomes

$$(\hat{K}\mu_1^{(j)})(x) = \frac{i\pi}{2}\int_{-\infty}^{\infty} (x-y)g(x, y, \lambda_j)u(y)\mu_0^{(j)}(y)dy. \tag{50}$$

In both Eqs. (48a) and (50), the solutions $\mu_0^{(j)}(x)$ and $\mu_1^{(j)}(x)$ are defined up to a multiplicative constant that can be determined using Eqs. (45a) and (46a).

Formulas (42), (44), (45a), (47) allow us to evaluate the limit of $c_j = -i(b_j/a_j') = -i(b(\hat{\lambda}_j)/a_{\zeta}(\hat{\lambda}_j))$.

$$\begin{aligned} b_j &= -\frac{1}{2i\delta\zeta_{-,j}}\int_{-\infty}^{\infty} u(y)\mathcal{M}_j(y)e^{-i\lambda_j y - \lambda_j \beta}dy \\ &= -\frac{1}{2i\lambda_j}\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)dy \\ &\quad + O(\delta^{-1}) = 1 + O(\delta^{-1}), \quad \delta \gg 1, \end{aligned} \tag{51}$$

so, as a consequence,

$$\mathcal{N}_j = \mathcal{M}_j/b_j = \delta\mu_0^{(j)}(x) + O(1), \quad \delta \gg 1. \tag{52}$$

For definition (9),

$$\begin{aligned} a_j' &= -\frac{1}{2i\delta\zeta_-^2}\int_{-\infty}^{\infty} u(y)\mathcal{M}_j(y)dy \\ &\quad + \frac{1}{2i\delta\zeta_-}\int_{-\infty}^{\infty} u(y)\mathcal{M}_j'(y)dy. \end{aligned} \tag{53}$$

$\mathcal{M}_j'(y) = \mathcal{M}_-(y, \zeta_-(\lambda))|_{\lambda=\lambda_j}$ satisfies the equation

$$\begin{aligned} \mathcal{M}_j'(x) - \int_{-\infty}^{\infty} G_-(x, y, \hat{\lambda}_j)u(y)\mathcal{M}_j'(y)dy \\ = \int_{-\infty}^{\infty} G'_{-,j}(x, y)u(y)\mathcal{M}_j(y)dy, \end{aligned} \tag{54}$$

with $G'_{-,j}(x, y) = G_{-,\zeta}(x, y, \zeta_-(\lambda))|_{\lambda=\lambda_j}$. The asymptotics of $G'_{-,j}$ (see Appendix B)

$$G'_{-,j}(x, y) = g_\lambda(x, y, \lambda_j) + O(\delta^{-1}), \quad \delta \gg 1 \tag{55}$$

and the condition (D5) of the existence of solutions of equation (54) for large δ, suggest the following ansatz:

$$\mathcal{M}_j'(x) = \delta^2\tilde{\mu}_0(x) + \delta\tilde{\mu}_1(x) + O(1), \quad \delta \gg 1. \tag{56}$$

Substituting (56) into (54), using (55) and (49c) we get, for the first two orders in δ,

$$(K\tilde{\mu}_0^{(j)})(x) = 0, \tag{57a}$$

$$\begin{aligned} (K\tilde{\mu}_1^{(j)})(x) &= \int_{-\infty}^{\infty} g_\lambda(x, y, \lambda_j)u(y)\mu_0^{(j)}(y)dy \\ &\quad + \int_{-\infty}^{\infty} g_1(x, y, \lambda_j)u(y)\tilde{\mu}_0^{(j)}(y)dy. \end{aligned} \tag{57b}$$

$\tilde{\mu}_0^{(j)}(x)$ is the solution of the homogeneous equation (24) with eigenvalue λ_j, then $\tilde{\mu}_0^{(j)} = \mu\mu_0^{(j)}(x)$, μ constant. Using this relationship, together with (49b), (28), and (D5), one gets $\mu = i/\pi$; then,

$$\begin{aligned} a_j' &= \frac{\delta}{2\pi\lambda_j}\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)dy + O(1) \\ &= -\frac{i\delta}{\pi} + O(1), \quad \delta \gg 1, \end{aligned} \tag{58}$$

and finally,

$$c_j = \pi/\delta + O(\delta^{-2}), \quad \delta \gg 1. \tag{59}$$

C. The inverse scattering problem

We are now ready to take the limit $\delta \to \infty$ of Eq. (13) which is the inverse scattering scheme for the ILW equation.

Let us analytically extend Eq. (13) to $\lambda = -\hat{\lambda}_j$:

$$\begin{aligned} &\left(e^{-i\lambda_j x - \lambda_j \beta} + \frac{iC_j}{\zeta_{-,j} + \zeta_{+,j}}\right)\mathcal{N}_j(x)_- \\ &\quad \times \frac{1}{2\pi i}\int_{-1/2\delta}^{\infty} \frac{\rho(\zeta'_-)\mathcal{N}(x, \zeta'_-)}{\zeta'_- + \zeta_{-,j} + i0}d\zeta'_- \\ &\quad + i\sum_{\substack{l=1 \\ l \neq j}}^{n} \frac{C_l\mathcal{N}_l}{\zeta_{-,l} + \zeta_{+,l}} = 1. \end{aligned} \tag{60}$$

Its limit when $\delta \to \infty$, evaluated using (42), (47), (59), (41), and (35), is

$$-\pi[x+\gamma_j(t)]\mu_0^{(j)}(x)-\frac{1}{2\pi i}\times\int_0^{\infty}\frac{\beta(\lambda)n(\lambda)}{\lambda-\lambda_j}\,d\lambda-i\pi\sum_{\substack{l=1\\ l\neq j}}^{n}\frac{\mu_0^{(l)}(x)}{\lambda_l-\lambda_j}=1, \tag{61}$$

where

$$\gamma_j(t)\doteq-\tilde{C}_j(t)/\pi+1/2\pi\lambda_j-i/2\lambda_j, \tag{62}$$

and $\tilde{C}_j(t)$ characterizes the second term of the expansion of C_j for Large δ:

$$C_j(t)=(\pi/\delta)(1+\tilde{C}_j(t)/\delta+O(\delta^{-2})),\quad \delta\gg 1. \tag{63}$$

Asymptotically in x, Eq. (61) reads $-\pi x\mu_0^{(j)}(x)=1$, then

$$\mu_0^{(j)}(x)=-1/\pi\Phi_j(x), \tag{64}$$

where $\Phi_j(x)$ is the solution of (24) with the property

$$x\Phi_j(x)\xrightarrow[x\to\infty]{}1.$$

So in terms of $\Phi_j(x)$, Eq. (61) becomes Eq. (31) and Eq. (45a) becomes

$$\int_{-\infty}^{\infty}u(y)\Phi_j(y)dy=2\pi i\lambda_j. \tag{65}$$

Let us consider now $\lambda\in\mathbb{R}^+$; in this case Eq. (13) goes directly to

$$\bar{n}(x,\lambda)-\frac{1}{2\pi i}\int_0^{\infty}\frac{\beta(l)n(x,l)}{l-(\lambda-i0)}\,dl=1-i\sum_{m=1}^{n}\frac{\Phi_m(x)}{\lambda-\lambda_m} \tag{66}$$

(see Appendix C), which implies that

$$-\frac{1}{2\pi i}\int_0^{\infty}\frac{\beta(l)n(x,l)}{l+i\epsilon}\,dl=1+i\sum_{m=1}^{n}\frac{\Phi_m(x)}{\lambda_m}. \tag{67}$$

The last choice for λ is $\lambda\in\mathbb{R}^-$; in this case,

$$n(x,-\lambda)e^{i\lambda x}-\frac{1}{2\pi i}\int_0^{\infty}\frac{\beta(l)n(x,l)}{l+i\epsilon}\,dl$$
$$-2\pi i\int_{-\infty}^{0}\frac{lf(-l)\bar{n}(x,-l)}{-le^{2l\delta}+\lambda e^{2\lambda\delta}+i\epsilon}\,e^{ilx+2l\delta}dl$$
$$=1+i\sum_{m=1}^{n}\frac{\Phi_m(x)}{\lambda_m}+O(\delta^{-1}),\quad \delta\gg 1.$$

Making use of (67), splitting $\int_{-\infty}^{0}dl$ into $\int_{-\infty}^{-\lambda}dl+\int_{\lambda}^{0}dl$ and expanding the corresponding integrands, we get

$$n(x,\lambda)e^{-i\lambda x}-\int_0^{\lambda}f(l)\bar{n}(x,l)e^{-ilx}dl=0,\quad \lambda>0 \tag{68}$$

(see Appendix C), which is nothing but the integral form of the analytic connection formula (27).

Formulas (66) and (68) are equivalent to the integral equation (30) and together with Eq. (31) they determine, in principle, the $n(x)$, $\bar{n}(x)$, and Φ_j's, and thus contain all the information one needs to solve the inverse scattering problem associated with the BO equation.

It is remarkable that these three equations are derived from the same Eq. (13) when $\delta\to\infty$, in the three different situations $\lambda=-\lambda_j$, $\lambda\in\mathbb{R}^+$, and $\lambda\in\mathbb{R}$.

Finally, the limit of (15) goes directly to formulas (33), showing how to reconstruct $u^+(x)$, and then $u(x)=u^+(x)-(u^+(x))^*$, the solution of the BO equation, from the scattering data.

D. Time evolution

In order to obtain the time evolution of the scattering data, we notice first of all that $\hat{\lambda}_j(t)=\hat{\lambda}_j(0)$, $j=1,2,\ldots,n$, then $\lambda_j(t)=\lambda_j(0)$ too. Moreover from (16), we get $C_j(t)=C_j(0)(1-2\pi\lambda_j t/\delta+O(\delta^{-2}))$, $\delta\gg 1$, while, from (63) at $t=0$, we get $C_j(0)=(\pi/\delta)(1+\tilde{C}(0)/\delta+O(\delta^{-2}))$, $\delta\gg 1$.

Comparing these results with formula (63), we infer the time evolution of $\tilde{C}_j(t)$ and, through (62), the time evolution of $\gamma_j(t)$:

$$\gamma_j(t)=\gamma_j(0)+2\lambda_j t,\ \gamma_j(0)\doteq-\tilde{C}_j(0)/\delta+(1/2\pi\lambda_j)(1-i\pi). \tag{69}$$

$\beta(\lambda,t)$ and $f(\lambda,t)$ originate from two different limits ($\lambda>0$ and $\lambda<0$) of $\rho(\lambda,t)$. Comparing the limit of Eq. (16),

$$\rho(\lambda,t)=\rho(\lambda,0)e^{i\lambda^2 t}(1+O(\delta^{-1})),\lambda\gtrless 0,\quad \delta\gg 1,$$

with formulas (41a) and (41b), we infer that

$$\beta(\lambda,t)=\beta(\lambda,0)e^{i\lambda^2 t},\quad f(\lambda,t)=f(\lambda,0)e^{i\lambda^2 t} \tag{70}$$

APPENDIX A

In this Appendix we will derive formulas (13) and (15) that characterize an alternative approach for solving the inverse problem of the ILW equation, to that given in Ref. 18, which is in terms of a Gelfand–Levitan– Marchenko equation. While the two approaches are equivalent for the ILW equation, it turns out that the one presented here is the most appropriate to describe the limit to the BO equation. Let us divide the scattering equation (8) by $a(\zeta_+)$; the function $M(\zeta_+)/a(\zeta_+)$ is analytic in the upper half ζ_+ plane except for poles (the zeros ζ_{+_j} of a); then

$$\frac{M(x,\zeta_+)}{a(\zeta_+)}=1+\mu_+(x,\zeta_+)+i\sum_{j=1}^{n}\frac{C_n N_j}{\zeta_+-\zeta_{+_j}}, \tag{A1}$$

where $\mu_+(x,\zeta_+)$ is analytic in the upper ζ_+ half plane and N_j and C_j are defined in (10b) and (14), respectively.

Expressing $\theta(\zeta_+ +1/2\delta)\rho(\zeta_+)N(x,\zeta_+)$ in terms of its $(+)$ and $(-)$ parts,

$$\theta(\zeta_+ +1/2\delta)\rho(\zeta_+)N(x,\zeta_+)=U^+(x,\zeta_+)-U^-(x,\zeta_-), \tag{A2}$$

$$U^{\pm}(x,\zeta_+)=\frac{1}{2\pi i}\int_{-1/2\delta}^{\infty}\frac{\rho(\zeta'_+)N(x,\zeta'_+)}{\zeta'_+-(\zeta_+\pm i0)}\,d\zeta'_+, \tag{A3}$$

and substituting all of this information into (8), we get $\mu_+(x,\zeta_+)=U^+(x,\zeta_+)$, and Eq. (13).

Equation (15) is obtained by considering Eq. (13) for large ζ_+. In order to do that, we must evaluate the asymptotics of $\bar{N}(x,\zeta_+)$ for large ζ_+.

$$G_-(x,y,\zeta_+)=\frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth\left[\frac{\pi(y'-x-i0)}{2\delta}\right]\times\left(\frac{1}{2\pi}\int_C\frac{e^{ip(y'-y)}(1-e^{-2p\delta})}{p-(\zeta_+ +1/2\delta)(1-e^{-2p\delta})}\,dp\right)dy'$$
$$=-\frac{1}{4i\delta\zeta_+}\coth[(\pi/2\delta)(y-x-i0)](1+O(\zeta_+^{-1})),$$
$$\zeta_+\gg 1. \tag{A4}$$

Then, using Eq. (6b), we get $\overline{V}(x, \zeta_-) \sim 1 - (1/\zeta_-)u^-(x)$ as $\zeta_- \to \infty$, and Eq. (13) yields Eq. (15).

APPENDIX B

We will briefly discuss here the procedure used to evaluate certain asymptotic calculations. As a prototype example, consider the integral

$$G_\pm(x, y, \lambda) = \frac{1}{2\pi}\int_{C_\pm} \frac{e^{i(x-y)p}}{p - (\zeta_+ + 1/2\delta)(1 - e^{-2p\delta})}\,dp. \tag{B1}$$

Using Cauchy's theorem, we may evaluate the order of magnitude of the contributions about 0 and λ, the two singularities of the integrand. Asymptotically in δ they are $(i/2)e^{i\lambda(x-y)}$ and $-i/4\lambda\delta$, respectively. Then we split the integral $\int_{-\infty}^{\infty} dp = \int_{-\infty}^{0} dp + \int_{0}^{\infty} dp$ and we expand the corresponding integrands; the first term gives a $O(\delta^{-1})$ contribution and the second one gives $g_\pm(x, y, \lambda)$. So

$$G_\pm(x, y, \lambda) = g_\pm(x, y, \lambda) + O(\delta^{-1}), \quad \delta \gg 1. \tag{B2}$$

Exactly the same procedure yields formula (55) and

$$G_{\pm\lambda}(x, y, \lambda) = g_{\pm\lambda}(x, y, \lambda) + O(\delta^{-1}), \quad \delta \gg 1, \tag{B3}$$

used in (36). The evaluation of $G_\pm(x, y, -\lambda)e^{i\lambda(x-y)}$ in (36) requires more attention:

$$G_{\pm\lambda}(x, y, -\lambda)e^{i\lambda(x-y)} = \frac{-2\lambda\delta + 1 - e^{-2\delta\lambda}}{2\pi(e^{\lambda\delta} - e^{-\lambda\delta})^2}$$
$$\times \int_{C_\pm} \frac{e^{ip(x-y)}(1 - e^{-2\delta(p-\lambda)})}{[p - (\zeta_+ + 1/2\delta)(1 - e^{-2p\delta})]^2}\,dp$$
$$= \frac{\lambda\delta}{\pi}\int_{C_\pm} \frac{e^{ip(x-y)-2p\delta}}{[p - (\zeta_+ + 1/2\delta)(1 - e^{-2p\delta})]^2}$$
$$\times dp(1 + O(\delta^{-1})). \tag{B4}$$

Replacing $\zeta_-(\lambda) + 1/2\delta$ with λ (with an exponentially small error) and rescaling p with $p\delta$ we finally get

$$G_{\pm\lambda}(x, y, -\lambda)e^{i\lambda(x-y)}$$
$$= \frac{1}{\pi\lambda}\int_{C_\pm} \frac{e^{-2p}}{(1 - e^{2p})^2}\,dp(1 + O(\delta^{-1}))$$
$$= -\frac{1}{2\pi\lambda} + O(\delta^{-1}). \tag{B5}$$

The evaluation of $G_-(x, y, \lambda_j)$ up to terms of order δ^{-1} is performed as follows:

$$G_-(x, y, \lambda_j) = \frac{1}{2\pi}\left(\frac{\pi i}{1 - 2\delta(\zeta_{-j} + 1/2\delta)}\right.$$
$$\left. + \int_{-\infty}^{\infty} \frac{e^{ip(x-y)}}{p - (\zeta_{-j} + 1/2\delta)(1 - e^{-2p\delta})}\,dp\right)$$
$$= -\frac{i}{4\lambda_j\delta} + \frac{1}{2\pi}\int_0^\infty \frac{e^{ip(x-y)}}{p - \lambda_j}\left(\frac{\nu_{-j} + \frac{1}{2}}{\delta(p - \lambda_j)}\right.$$
$$\left. + \sum_{n=0}(-1)^n \frac{\lambda_j^n e^{-2np\delta}}{(p - \lambda_j)^n}\right)dp$$
$$+ \frac{1}{2\pi\lambda_j}\int_{-\infty}^{0} e^{ip(x-y)+2p\delta}$$
$$\times\left(\sum_{n=0} \frac{(-1)^n(p - \lambda_j)^n e^{2np\delta}}{\lambda_j^n}\right)dp + O(\delta^{-2})$$
$$= g(x, y, \lambda_j) + (1/\delta)g_1(x, y, \lambda_j) + O(\delta^{-2}), \quad \delta \gg 1, \tag{B6}$$

APPENDIX C

In this Appendix, we will discuss the asymptotic behavior of $n(x,\lambda)$, $m(x,\lambda)$, $\bar{n}(x,\lambda)$ when $\lambda \to 0$. The same ideas will also be used to obtain equations (43)–(45).

Let us consider function $n(x, \lambda)$, solution of

$$n_x - i\lambda n = i[un]^+, \quad n \xrightarrow[x\to-\infty]{} e^{i\lambda x} \tag{C1}$$

($[h]^+$ indicates the $(+)$ projection of h), or, equivalently, the solution of

$$n(x, \lambda) = e^{i\lambda x} + \int_{-\infty}^{\infty} g_-(x,y,\lambda)u(y)n(y,\lambda)\,dy. \tag{C2}$$

Noticing that $g_-(x, y, \lambda) \sim -1/2\pi \ln \lambda$ as $\lambda \to 0$, Eq. (C2) will be satisfied at the $O(1)$ iff

$$n(x, \lambda) \sim n_0(x)/\ln \lambda, \quad \lambda \to 0, \tag{C3}$$

where $n_0(x)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty} u(y)n_0(y)\,dy = 2\pi. \tag{C4}$$

Substituting (C3) into (C1), we get

$$n_{0_x} = i[un_0]^+. \tag{C5}$$

Then Eq. (C5) and the normalization condition (C4) define the coefficient $n_0(x)$ of the leading term in the asymptotic expansion of $n(x, \lambda)$ when $\lambda \to 0$. In particular, it is easy to show that (C5) and (C4) imply that $n_0(x) \xrightarrow[|x|\to\infty]{} -\ln x$.

In exactly the same way, it is possible to show that

$$m(x, \lambda), \bar{n}(x, \lambda) \sim n_0(x)/\ln \lambda, \quad \text{as } \lambda \to 0. \tag{C6}$$

Moreover, using (C6) and (C4), we can easily get

$$\beta(\lambda) \sim 2\pi i/\ln \lambda, \quad f(\lambda) \sim -1/\lambda \ln \lambda, \quad \lambda \to 0. \tag{C7}$$

Formulas (C7), as well as (C3) and (C6), are implicitly used to prove the validity of (67) and to show that the integrals contained in formulas (66) and (68) are well defined.

Formulas (C3)–(C6) supercede the formulas (24) in Ref. 18 (the first of which is incorrect; however, only the or-

der of magnitude of the limit $\lambda \to 0$ was used in Ref. 18. This indicates that $\int_{-\infty}^{\infty} u\,dx = 0$ is not special in the limit $\lambda \to 0$.

Let us now prove by contradiction that $\alpha_0^{(j)} = 1$. Then let us suppose that $0 < \alpha_0^{(j)} < 1$; it follows that

$$\zeta_{+}(\hat{\lambda}_j) = (\hat{C}_j - \tfrac{1}{2})\frac{1}{\delta} + O(\delta^{-2}); \quad \hat{C}_j \doteq \frac{\pi \alpha_0^{(j)} e^{i\pi\alpha_0^{(j)}}}{2\sin(\pi\alpha_0^{(j)})},$$

and

$$G_{+}(x, y, \hat{\lambda}_j) \sim \frac{1}{2\pi}\int_{C_+} \frac{e^{i(x-y)p}}{p - (\hat{C}_j/\delta)(1 - e^{-2p\delta})}\,dp$$
$$\sim -\frac{1}{2\pi}\ln\left(\frac{\hat{C}_j}{\delta}\right).$$

Consequently using the same kind of arguments leading to (C3) and (C4), one can show that Eq. (6a) implies

$$M_j(x, \delta) \sim \frac{M_{0j}(x)}{\ln(\hat{C}_j/\delta)}, \quad \text{with} \int_{-\infty}^{\infty} u(y) M_{0j}(y)\,dy = 2\pi.$$

Then

$$a_j \sim 1 - \frac{i\pi}{(\hat{C}_j - \frac{1}{2})\ln(\hat{C}_j/\delta)} \sim 1 \neq 0,$$

which contradicts the hypothesis.

Analogously, if $\alpha_0^{(j)} = 0$, $\zeta_{+}(\hat{\lambda}_j) = (i\pi/2\delta^2)\alpha_1^{(j)} + O(\delta^{-3})$ and $G_{+}(x, y, \hat{\lambda}_j) \sim -(1/2\pi)\ln(1/2\delta)$. Consequently Eq. (6a) implies

$$M_j(x,\delta) \sim \frac{M_{0j}(x)}{\ln(1/2\delta)}, \quad \text{with} \int_{-\infty}^{\infty} u(y) M_{0j}(y)\,dy = 2\pi.$$

Then

$$a_j \sim 1 - \frac{2\delta}{\alpha_1^{(j)}\ln(1/2\delta)} \sim -\frac{2\delta}{\alpha_1^{(j)}\ln(1/2\delta)} \neq 0,$$

which again contradicts the hypothesis. So we are left with the only choice $\alpha_0^{(j)} = 1$. In this case,

$$\zeta_{-j} = \lambda_j + \nu_{+j}/\delta + O(\delta^{-2}),$$

so

$$a_j = 1 + \frac{1}{2i\delta\lambda_j}\int_{-\infty}^{\infty} u(y) M_j(y)\,dy\,(1 + O(\delta^{-1}))$$

will be zero only if (44) and (45a) hold.

APPENDIX D

Given the following equation,

$$(Kh)(x) = 1 - C_1\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)\,dy$$
$$- C_2\int (x - y) u(y) g(x, y, \lambda_j) \mu_0^{(j)}(y)\,dy \doteq F(x) \qquad \text{(D1)}$$

[the operator K is defined in (49a)], Fredholm theory says that a solution exists iff

$$\int_{-\infty}^{\infty} \phi^*(x) F(x)\,dx = 0, \qquad \text{(D2)}$$

where ψ satisfies the equation $(K^{+}\psi)(x) = 0$, where K^{+} is the adjoint operator of K:

$$(K^{+}h)(x) = h(x) - u^*(x)\int_{-\infty}^{\infty} g^*(y - x, \lambda_j) h(y)\,dy. \qquad \text{(D3)}$$

As a consequence of the equation $(K^{+}\psi)^*(x) = 0$, we have that

$$\int_{-\infty}^{\infty} \psi^*(x) F(x)\,dx = \left[1 - C_1\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)\,dy\right]\int_{-\infty}^{\infty} \psi^*(x)\,dx. \qquad \text{(D4)}$$

Then the condition $\int_{-\infty}^{\infty} \psi^*(x)\,dx \neq 0$ implies

$$1 + C_1\int_{-\infty}^{\infty} u(y)\mu_0^{(j)}(y)\,dy = 0. \qquad \text{(D5)}$$

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